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THE ROLE OF SYMBOLS IN MATHEMATICS LEARNING

BY

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DAVID J. BALE

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
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FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "The Role of Symbols in Mathematics Learning" submitted by David J. Bale in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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Abstract

The purpose of this study was to develop and compare three approaches to the role of symbols in the learning of structural relations of mathematics. All three approaches were based on Dienes' theory but differed in the rôle assigned to symbols in the learning. These approaches were developed into treatments whereby the defining properties or axioms of the mathematical group could be taught to grade seven students over a four week period. The Concrete treatment followed Dienes' ideas as closely as possible. In phase I, the structural relations, the properties of the mathematical group, were learned by manipulation of physical objects or their mental images. Symbols were introduced gradually only after the properties had been induced. Phase I of the Primary Symbolic treatment called for learning in the context of familiar symbols, meaningful to the learner. For the Secondary Symbolic treatment, phase I consisted of learning in the context of initially meaningless symbols. Phase II of each treatment consisted of a brief presentation of one embodiment from each of the other two treatments.

The effects of these treatments on symbolization, interpretation, transfer and application of the defining properties of the mathematical group for the grade seven students was determined by multivariate analyses of variance on the treatment mean vectors. The relationship of reflective intelligence, arithmetic achievement and sex to the four criterion outcomes, and interaction of these learner characteristics with the treatments was also studied.

There was no significant difference among treatments at the .05 level, but there was at the .08 level. Further univariate analysis of variance indicated that the difference with the lowest probability was in favor of the Secondary Symbolic treatment over the Primary Symbolic treatment on the Interpretation test. A significant difference at the .05 level favoring the high student groups occurred for reflective intelligence on Symbolization and Interpretation, and for Arithmetic Achievement on these two tests and Transfer. Though the group mean vector comparison favored girls over boys at the .05 level, this did not show up on any individual tests. There was no interaction of these learner characteristics with treatments.

While the evidence is not strong enough to suggest that any one of the three treatments is superior to the other two, it is sufficient enough to warrant a call for further investigation of the effect of the role of symbols on symbolization, interpretation and transfer of structural relations, particularly with respect to the connection between these processes and abstraction. Meanwhile, the study shows that success in learning the group structure properties is related to reflective intelligence and arithmetic achievement, regardless of treatment.

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Though the experience has been rewarding at times it has seemed a long and tortuous road to the completion of my graduate studies. Now that an end is in sight, I would like to express my appreciation to the many people who have helped to make this accomplishment possible.

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I appreciated the opportunity to conduct the experiment at St. Cecilia school, and wish to express my thanks in particular to the grade seven students who participated and to Mr. Ray Baron, one of their teachers.

Finally, I dedicate this study to my family, Judy, Marty, Nancy and Kelly for their love, patience and faith in me.

TABLE OF CONTENTS

CHAPTER PAGE

I THE PROBLEM 1

General Statement of the Problem. 2

Background to the Study 3

Significance of the Study 7

Outline of the Report 7

II CONCEPTUAL FRAMEWORK AND RELATED STUDIES. . . . 9

Dienes' Theory of Optimal Learning of Structural Relations of Mathematics 9

Mathematics. 10

Thinking 14

Learning 19

Symbols. 26

Related Research. 29

III DESIGN OF THE STUDY 39

The Logic of the Study. 39

Three Alternative Approaches to the Role of Symbols 39

The Concrete Approach. 43

The Primary Symbolic Approach. . . . 44

The Secondary Symbolic Approach. . . 45

Treatments 46

Criterion Variables. 47

Learner Related Variables. 49

The Experimental Study. 50

Experimental Questions 50

CHAPTER PAGE

Experimental Procedures 51
The Sample. 51
Learner Related Measures. 52
Experimental Design 53
Null Hypotheses 56
IV THE DEVELOPMENT OF THE TREATMENTS AND CRITERION MEASURES 57
Introduction 57
The Treatments 58
The Concrete Treatment. 58
The Primary Symbolic Treatment. 69
The Secondary Symbolic Treatment. 72
The Criterion Measures 75
V THE RESULTS OF THE INVESTIGATION 79
Comparison of Treatments 80
Comparison of Categorization Variables 84
Interaction Effects 84
Categorization Variable Effects 85
Summary. 87
VI SUMMARY, CONCLUSIONS AND IMPLICATIONS. 89
Summary. 89
Conclusions. 90
Discussion. 91
Dienes' Most Recent Findings. 94
Limitations. 97
Implications 98
For Practice. 98

CHAPTER PAGE

For Theory 98

For Further Research 99

BIBLIOGRAPHY 101

APPENDIX A 106

APPENDIX B 116

LIST OF TABLES

Table		Page
I	Correlations Between All Possible Pairs of Criterion Tests	80
II	Group Concepts Tests Cell Means for the Three Treatments	81
III	Summary of Multiple Analysis of Variance on Treatment Mean Vectors, Rao's F-test	81
IV	Summary of Analyses of Variance for the Three Treatments on the Means of the Group Concepts Tests.	82
V	Tests of Differences Between Ordered Mean Scores for the Three Treatments on the Interpretation Test.	83
VI	Group Concepts Tests Cell Means for Each Category by Treatment.	85
VII	Summary of Rao's F-Test for Multiple Analysis of Variance on Category Main Effects and Treatment by Category Interaction.	86
VIII	Multiple Comparisons Using Rao's F-Test for the Three Categorization Variables on Group Concepts Tests	87

LIST OF FIGURES

Figure | Page

1 The Three Approaches on a Concrete Abstract Dimension 42

2 The Three 2 x 3 Factorial Designs of Learner Related Variables vs. Treatments. 55

3 The Happy Face Card 59

4 First Data Recording Chart. 63

5 Second Data Recording Chart 64

6 Lattice Table for a Secondary Symbolic Embodiment. 74

CHAPTER I

THE PROBLEM

"Queen and servant of science," "an abstract game," "a product of man's mind," these are but a few of the more colorful phrases used to describe mathematics. Such descriptions reflect the belief that mathematics involves much that is highly abstract. One aspect contributing greatly to this notion is the type of language used to communicate mathematics. To a large extent this language consists of a specialized kind of symbolism.

The importance of mathematical symbolism is pointed out by many, such as Katsoff (1949) by his comment, ". . . many have insisted that the fertility of mathematics is due to its symbols . . ." (p. 3), and by Whitehead (1967) in his remark, "By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race." (p. 39).

By way of contrast, the tendency of mathematical symbolism to heighten the impression of abstractness of the discipline has created learning problems. The National Council of Teachers of Mathematics (1959) has expressed its concern in such statements as, "mental confusion and fear are often the by-product of many students' first contacts with the symbols of mathematics" (p. 301), and:

> . . . any non-mathematician, on looking inside an advanced mathematics textbook will be horrified and will shut the book at once. The horror of alien symbols is due simply to the fact that most present day adults were never taught during their schooldays, how to decode mathematical symbolism. (p. 279).

These views support the need for close examination of the role of mathematical symbols in the learning of mathematics.

General Statement of the Problem

If we wish to develop students' abilities to handle mathematical symbols to the extent that they can make use of the power of the symbols to create and communicate mathematical ideas, then we must determine the role of the symbols in the learning process. This raises the questions of how and when to introduce symbols. What amount and kind, if any, of non-symbolic experiences involving the concepts will be needed during initial learning? Should symbols appear first during the learning phase, or only subsequently, can learning sequences reflecting the different approaches implied by these questions be constructed? To what extent will the reflective intelligence, degree of achievement in arithmetic, or sex of the student be related to the learning under any such approaches devised?

The aim of this study is to investigate these questions from a theoretical and a practical point of view. The first step is to examine a theory of mathematical learning which is highly related to these questions. The second step is to generate hypotheses about alternative approaches to the role of symbols. These hypotheses reflect concerns raised by the questions in the framework of the theory. The next step is to create treatments representing the alternative approaches, and the final step is to test these experimentally in a school setting. The main idea behind this study of symbols is to determine principles governing the role of symbols in learning mathematics that can be consistently built into the broader theory examined.

Background to the Study

The theoretical basis of the present study is of interest to educators primarily as a result of its relationship to two trends in New Mathematics which first appeared in the curricula of the sixties. The first of these two to emerge was emphasis on the structure of disciplines in school programs. Crawford (1966) attributes this to a shift in philosophical emphasis.

> It has been argued that two main aims for school mathematics should be:
>
> 1. to make students familiar with the global structure of mathematics, and
>
> 2. the inculcation of an understanding of what mathematics is (and what it is not). (p. 1).

According to Scott, (1966) this emphasis on structure resulted in a number of changes in school mathematics programs, most notably an earlier introduction of topics, greater abstraction, and refinement of nomenclature and symbolism.

The second important trend was a shift from teacher-oriented to child-centered programs. This was just beginning in the sixties and continued to expand during the early seventies in North America. According to Edith Biggs (1969), one of the leading proponents of the active learning approach,

> The aims of learning mathematics in this active, creative way are:
>
> 1. to free students, however young or old, to think for themselves,
>
> 2. to provide opportunities for them to discover the order, pattern and relations which are the very essence of mathematics, not only in the man-made world, but in the natural world as well,
>
> 3. to train students in the necessary skills. (p. 3).

Interest in active learning brought about renewed concern for individual differences among children in cognitive style, and a re-examination of learning contexts, particularly the concrete mode.

The activity oriented approach followed the trend to more structure and symbolism. The importance of the concrete mode was recognized in the design of many elementary mathematics programs of the early seventies. Before the trend to even more emphasis on the concrete mode, or a reversal back to the symbolic mode takes place, more knowledge of the role of symbols is necessary. Does emphasis on symbolism imply more power, but greater risk of learning difficulties? Is there a developmental factor which requires learning in concrete contexts at lower grade levels, but permits or requires learning in symbolic contexts at higher grade levels?

Among mathematics educators who have addressed themselves to such problems, Zoltan Dienes has come closest to grappling with the questions of this study. He advocates emphasis on structural relations and proposes unconventional, seemingly more abstract mathematics topics for the elementary school grades. However, in recognition of learning difficulties associated with mathematical symbols, he has developed concrete modes of presentation.

Dienes has postulated several principles governing optimal learning of mathematics derived primarily from his ideas on cognitive development and mathematical structure. The essence of those principles relating to the role of symbols in learning mathematics is that concrete experiences are necessary for abstraction of the concepts or relations involved, and that symbols may only be introduced after the abstraction has been made. Richard Skemp (1963) criticized the necessity of learning in concrete contexts. He maintains that the nature of most mathematical concepts requires that they be learned in symbolic contexts. These symbols represent previously learned mathematics, and are therefore considered semi-concrete. The new concepts are learned by reflecting on some particular aspects of this previously learned mathematics, and therefore requires a highly developed reflective intelligence. Reflective intelligence is positively correlated with age.

Dienes' concrete approach and a modified approach based on Skemp's criticism, called a Primary Symbolic approach, could serve as the basis for two treatments differing in the role assigned to symbols in learning. The concrete approach calls for relegating symbols to labels applied after learning of the concepts has taken place. The Primary Symbolic approach calls for symbols to call to mind the pre requisite knowledge on which the new concepts are based. The new concepts in turn are given new symbols tacked on after as labels. The symbols which serve as the basis of learning, the primary symbols, are semi-concrete, the symbols attached to the newly learned concepts also eventually take on a semi-concreteness and are available for new learning. Since these two approaches occupy the concrete and semi-concrete locations on a concrete-abstract dimension, a more abstract approach suggests itself as a third possibility. This approach, based on formalistic ideas of mathematics has been suggested by Dienes (1965) as a possibility worthy of study, and has been mentioned by Bruner in the preface to Dienes (1963) as a viable alternative. The central aspect of this approach is that learning takes place in the context of symbols for which no meaning other than abstract structural relations may be induced.

This study was initiated in order to compare these three approaches which differ in the role assigned to symbols. In addition, since learner factors such as reflective intelligence appeared to be related to the theory behind one or more of the approaches, the most significant of these factors were also examined in this study. Reflective intelligence and arithmetic achievement are included because they appear to be related to achievement under the primary symbolic approach. Sex is included because in Dienes' study, Concept Formation and Personality (1960), he found evidence of sex differences in learning structures.

Significance of the Study

A study of the role of symbols in the learning of mathematics is important from both a practical and theoretical point of view. Trends of the past few years indicate shifts from symbolic to concrete modes of presentation of mathematical content. Shifts back and forth are likely to continue until we can determine the most appropriate use of each of these modes. If this can be done within the framework of a theory of learning mathematics, such as that of Dienes, so much the better. The problem of this study may shed light on the role of symbols. Certain results of comparison of treatments and learner characteristics would lead to conclusions of direct application to Dienes' theory. For instance, if the concrete approach was found to be superior for all criterion outcomes and all types of subjects identified then Dienes' theory would be supported. However, some other kinds of outcomes would be less easy to interpret, and would lead to implications for further research.

Outline of the Report

Chapter II provides a theoretical background, primarily related to Dienes' works, and a review of the most relevant research on the problem of the study. A detailed description of the design of the study and the statistical procedures used is presented in Chapter III. In Chapter IV the development of the treatments and criterion tests is described, Chapter V consists of the presentation of the results of the statistical analysis and Chapter VI consists of conclusions and implications.

CHAPTER II

CONCEPTUAL FRAMEWORK AND RELATED STUDIES

Dienes' Theory of Optimal Learning of Structural Relations of Mathematics

For nearly two decades Zoltan Dienes has conducted practical classroom research in an attempt to build a theory of mathematics learning. Throughout Dienes has expressed concern over the role of symbols in the learning. His approach has been developmental with findings determining directions and modifying basic notions. He has, however, maintained a constant view of the nature of the problem. Dienes (1960) claims that,

> The problem of learning is essentially how to find a kind of 'best fit' between the structure of the task and the structure of the person's thinking. For the process to be explained by intelligable theory, both these structures must be taken into account. (p. 39).

These notions of structure of the task and the person's thinking are central to Dienes' theory. Through an analysis of his ideas on mathematics and the person's thinking, it is possible to see how he derived a model for optimal learning of mathematics, and postulated a role for symbols in the model. This model takes the form of a set of principles which are statements of conditions under which optimal learning of mathematics takes place. Richard Skemp has also carried out research in this area. Though his ideas resemble those of Dienes, they are in direct opposition on several crucial points, particularly with respect to the role of symbols.

This chapter consists of a discussion of the views of Dienes under the headings of Mathematics, Thinking, Learning, and Symbols. Skemp's disagreement is aired primarily in the latter section, though its basis is touched on in each of the preceding sections. In order to explain Dienes' position as clearly as possible, the views of Piaget, Bruner and Bartlett, on which he claims to have based his own ideas, are also mentioned.

Mathematics

According to Dienes (1960) and (1964), mathematics is a structure of relationships, the accent being on the structure more than on any content it might fit. He uses the term structure in a very broad sense. He speaks of a person sorting out environmental events by structuring them, and of teaching the structured stage. This suggests that he views structure as something created by thought. More specifically, we form classes of situations by thinking about our environmental observations in particular ways. Structures describe these classes of situations. In mathematics there are two types of structures, formal and informal. The content of abstract algebra consists of relationships within and among formal structures such as groups, fields and vector spaces. This is so because abstract algebra is the study of algebraic form. Dienes and Golding (1967) have provided copious concrete examples of formal structures such as groups.

Dienes (1964) gives an example of an informal structure as the use of factors of quadratic functions for the purpose of achieving the solution of a whole class of problems. The class of situations are all those problems involving quadratic functions for which factorization provides a solution. The factorization identity equation is the mathematical aspect of the structure which has been abstracted from a number of different situations.

Dienes (1961) feels that it is not meaningful to separate different branches of mathematics such as algebra and arithmetic because there is so much connection between them. Certain algebraic concepts, such as distributivity should be known before some arithmetic operations can be truly mastered. These concepts may consist of "classes of classes of classes of . . ." etc., of objects heaped together in all sorts of relationships to each other. Mathematicians have analyzed this maze of classes and organized it into structures.

In one of his books Dienes (1964) describes structure as relationships among the parts and between parts and whole. While he feels these abstract relationships may stand alone without a more concrete interpretation, they probably represent the distillation of past concrete experiences. Furthermore, the structures may serve as models to evaluate environmental events or phenomena. For these reasons Dienes views mathematics as the actual structural relationships among the concepts connected with numbers (pure mathematics), together with applications to problems arising in the real world (applied mathematics). He sees these two faces of mathematics as inseparable, and thus profoundly influencing the way mathematics is learned. On the one hand, pure mathematics emphasizes relationships within and among structures. If this is so, the ability to discern and transfer structural relationships would be important in the learning process. On the other hand, applied mathematics emphasizes concrete experience, so the ability to abstract classes from real objects or events, and the ability to apply structural relationships to the real world would also seem to be of prime importance.

The structures which mathematicians must consider are so complex, in Dienes' view, that it would be impossible to dispense with symbolism. He describes a symbol as a sign that reminds us of 'something' apart from itself. This 'something' gives meaning to the symbol. In the case of word signs, they call our attention to certain imagery which accompanies the classes of things the words symbolize. The symbolic expression of mathematics is a language in terms of which we speak of structures. Thus symbols express both abstract and concrete contents. The abstract content is the structural relationships and is expressed by combining several symbols into a mathematical statement. This is known as the symbolization process. The concrete content is an experiential interpretation (manipulations of real objects or events) of the structural relations in which each individual symbol stands for a concrete entity, manipulation or relation. Identifying a concrete embodiment of the symbolized structure is known as interpretation.

The following statement of Dienes (1963) on the relationship of symbolization and interpretation to each other provides clues to his views on these two processes: " . . . unless both processes, structure to symbol (symbolization) and symbol to structure (interpretation) can be carried out easily, the symbols are not yet doing their job." (p. 131). Obviously, Dienes is stressing the abstract aspect of symbols here, in which symbols convey structural relations, not concrete content. It is not that the symbols are abstract, only that they symbolize something abstract. This 'something abstract' is the structural relations which have been abstracted as the common essence of a number of different situations. Hence symbolization and interpretation are more than just coding and decoding involving concrete content represented by symbols. In Dienes' view, symbolization of mathematics involves formulating statements about abstract structural relations. Interpretation is non-symbolic demonstration of structural relations in terms of embodiments of the structure of which they are a part.

The disagreement between Skemp and Dienes centers around the physical world aspect. Skemp (1963) says, "Mathematics is not a collection of facts which can be demonstrated and verified in the physical world, but a structure of closely related concepts, arrived at by the process of pure thought." (p. 41). Thus he appears to be in complete disagreement with Dienes regarding applied mathematics. Skemp classifies only the structural relations aspect as mathematics. Much of what is generally referred to as arithmetic is not classified as mathematics by Skemp, as concepts of number, addition, and so on are derived from direct sensory experience rather than pure thought. Such concepts, which Skemp refers to as primary concepts then, become available as objects of thought in building mathematical concepts. The mathematical concepts are referred to as secondary concepts which are derived from primary or other secondary concepts. With respect to symbols Skemp believes that whereas primary concepts can be exemplified by physical objects, secondary concepts can only be symbolized. It appears that Dienes and Skemp would not disagree over the structural relationship aspect that symbolization represents. However, there would be considerable disagreement over the meaning of symbols. Whereas in Dienes' view, all mathematical symbols express concrete content, only primary symbols do so directly in Skemp's view, and secondary symbols express abstract concepts which are traceable eventually through a hierarchy of concepts to concrete content. But this concrete content, once reached, is devoid of the structural relationships. This is so because structural relationships are derived from objects of thought, not physical objects.

Thinking

Dienes (1960) has stated that Bruner and Bartlett provided the basis for many of his ideas on thinking. When Bruner (1966) set out to study the ways in which people think, he discovered several characteristic strategies. This led him to certain conclusions concerning cognition and the nature of its growth. He felt that some of the strategies could be systematically described and evaluated. Coupled with other considerations, this knowledge led him to postulate the existence of cognitive structures called generic coding systems, and three stages of growth in the representation of thought, namely the enactive, iconic, and symbolic.

The strategies were categorized into two main types, analytic and intuitive. Although Bruner found it difficult to recognize the intuitive process he was able to describe it by means of a working definition as immediate apprehension or cognition, and proceeded to describe the process in general terms as follows:

> In contrast to analytic thinking, intuitive thinking characteristically does not advance in careful, well defined steps. Indeed, it tends to involve maneuvers based seemingly on an implicit perception of the total problem. The thinker arrives at an answer, which may be right or wrong, with little if any awareness of the process by which he reached it. (p. 241).

Dienes postulated parallel categories to these which he calls constructive and analytic. He believes children's thought in mathematics is largely of the constructive kind, and furthermore that people vary in the degree to which they use one or the other type of thinking. Generally speaking, even for adults mature in a particular field of thought, optimal thought demands that construction precede analysis.

Concerning productive mathematical thought Dienes believes that the preliminary constructive stage is always operative. In his view, mathematical thinking is a cyclic process which can be divided into the two distinct stages, constructive and analytic. A paraphrasing of Dienes' (1960) description of how these two stages operate in the process of mathematical thinking is as follows. In the constructive stage the mathematician is given a structure which takes him so far and no further. He then proceeds with randomly trying this and that until eventually a more purposeful build-up occurs. Subsequently the evidence is sorted out and organized into initial patterns and a kind of direction is discerned. Guided by a 'hunch', he follows this direction which leads him to create some sort of provisional structure. Thus the initial phase of the cycle is completed, and the work of analysis begins. The mathematician now reflects on what he has constructed; he carefully and critically examines any relevant class relationships. There is a natural tendency to establish a kind of enclosure around the newly analyzed structure but the mathematician adopts a strategy of endless open thought moving from construction to analysis to construction, and so on.

> This kind of patterning is, on the one hand, the very essence of mathematical thinking. Moreover, established patterns soon come to be regarded as mathematical objects, which are fitted into further patterns; these in turn, upon becoming familiar are regarded as objects,

and so on. (pp. 32-33).

Thus the cyclic nature of the process comes into play. Bartlett (1958) also speaks in terms of intuitive and analytic type thinkers, but establishes an additional category, a mixture of the analytic and intuitive. In this mixture intuition and analysis alternate back and forth until a conclusion is reached.

Dienes, relying heavily on Bruner's and Bartlett's ideas, and his own notion of how a mathematician thinks, has derived two principles, the Dynamic Principle and the Constructivity Principle which form part of his theory of optimal learning of mathematics. In the Dynamics Principle, in place of the final analytical stage in the cycle, Dienes has substituted an alternative, practice. Whereas the mathematician engages in open search, one structure leading to another, Dienes feels most children are capable of very little analysis, so for them construction leads to practice, leads to further construction, and so on, with very little analysis occurring at all.

It is at the practice stage that symbols, according to Dienes (1964), enter the picture. One function of the symbols is "to fixate the realization of the extent to which a certain structure will reach." (p. 140). In other words, the symbolization serves to recall abstracted or generalized structural relation to one's mind. The symbolism also may then act as an embodiment of the structure, and may be used to process information and generate new structures. Of course, this latter aspect is operative only at the analytic stage, so little in this way would be accomplished by very young children in Dienes' view.

> In the average mathematical-learning situation the thread between reality and the manipulation of 'symbols' is very thin, if not definitely severed. Even in situations based on concrete mathematical experiences, this thread may break. Either the teacher or the student becomes over enthusiastic about the formal properties of the rule-structures being unravelled through mathematical experiences, and loses touch with reality. Constant feedback, or at least the opportunity of feedback, into concrete situations from which the structures have been abstracted will guard against this danger. (p. 143).

Losing sight of the concrete content of the symbols devoids them of mathematical meaning according to Dienes' notion of mathematics. So while this formal playing about might appear to be mathematical analysis, Dienes would not classify it as such unless the structural relations derived could be interpreted in terms of their concrete content, or applied to situations in the physical world.

The significant contribution of Skemp to the area of mathematical thinking has been his work on reflective intelligence, the ability of the mind to become aware of, and manipulate, its own concepts. This type of thinking is essential to his theory of learning in view of his interpretation that mathematics is a pure construction of the mind. Skemp (1963) refers to organized structures of knowledge as schemas, and the kind of learning that makes use of, and builds more knowledge onto this structure is called schematic learning. This is similar to Dienes' notion of one structure leading to another. However, whereas

Dienes sees the process as taking place in a cyclic way at an entirely constructive level, or alternating back and forth between construction and analysis, for Skemp the process is entirely analytical. The new structure is being learned by reflecting on a symbolic representation of a previously learned structure. Thus the roles of symbols in thinking envisioned by Dienes and Skemp are entirely different from each other. For Skemp, symbols play a central role in learning mathematics, but for Dienes they assume importance only after the structural relations have been abstracted through constructive activity.

Learning

Although Dienes' learning theory can be traced to his views on the nature of mathematics and mathematical thinking, it also derives from the cognitive development theories of Piaget and Bruner. These two theories are quite similar in that both adhere to the notion of stages of development. Piaget's stages are well known. The initial stage of interest to Dienes is the pre-operational one in which thought can be interpreted logically in only very simple situations. In the next stage, concrete operations, logic is developing, but it is still tied to concrete experience. In the final stage, that of formal operations, logic becomes fully operational and independent of concrete experience. Bruner's enactive, iconic and symbolic levels of representation of thought are similar to Piaget's stages. However, the stages are not necessarily tied to a level of growth as an adult might pass through all three stages in learning a new concept. Dienes (1961) subscribes to this Brunerian notion as evidenced by the following statement.

> It is suggested that what Piaget perceived as a developmental cycle in the large, as a macrocosmos as it were, also occurs in the formation of every abstract concept as a microcosmos. But not only does this happen in adult life, it happens in childhood, the structure of the cycles being the same, only the content being different. (p. 283).

The aim of learning mathematics in Dienes' view is the comprehension of structural relationships between mathematical concepts, and the acquisition of the ability to apply the resulting concepts to situations occurring in the world. These concepts and relationships are learned by the processes of abstraction and generalization.

> The process of abstraction is defined as the process of drawing from a number of different situations something which is common to them all. . . . This fundamental relationship between classes and their elements, i.e., the relationship of 'belonging' or 'being an element of' is realized in the direction from elements to class. The process of generalization, instead of leading from elements to classes leads from classes to classes. (p. 281-82).

Dienes distinguishes two types of generalization, primitive and mathematical. Primitive generalization is passing from one class to a second which includes the first as a part of it. Mathematical generalization is similar but the second class includes only an isomorphic image of the first class, not the actual class itself. For example, passing from the natural numbers to the integers which include an isomorphic image of the natural numbers is an example of mathematical generalization. Dienes does not go far enough in his description of abstraction. It is not clear whether or not he would subscribe to the notion of degrees of abstraction. The indications in his writings are that he has not accounted for the possibility of degrees of abstraction. He recognizes that irrelevancies in a situation may interfere with an abstraction, and that some people are better than others at abstracting, but feels that providing more learning experiences is the solution to both these problems. He conveys the impression that he believes that once a class has been abstracted it is known, and the person who knows the class will be able to recognize all exemplars and non-exemplars. One who cannot recognize exemplars and non-exemplars as such does not know the class. He has therefore not abstracted the class and needs a greater variety of experiences embodying the structure in order to abstract it.

Concept attainment, in Dienes' view, begins with the play stage. During this period the occurrence of undirected activity, seemingly purposeless, in which there is freedom to experiment, is in evidence. The next stage, passing from apparent chaos to structure, from apparent lack of direction to the realization of direction in one's thinking, by means of the process of concept formation, corresponds to the kind of learning that takes place during the concrete operations stage as described by Piaget. Dienes believes that concrete objects or mental images, isomorphic to structures with regard to constituent relationships, are being put together according to some plan during this stage. Dienes' third stage is that of logical analysis - reflecting on what one has done and discerning how it is really put together. The final stage, that of practice, consists of making use of what has been done.

This natural process of concept formation fits the model of thought representation proposed by Bruner in that the play stage might parallel the enactive representation to some extent, and this would be continued in the concept forming stage along with iconic type representation. Symbolic representation would not likely be evident until Dienes' third stage, that of logical analysis. Also, this concept formation cycle postulated by Dienes, in his view, has much in common with the thought processes involved in the search for mathematical discoveries by mathematicians. However, for the concrete operations child, the final stages consist of practice, and very little of logical dissection; as he grows older, the child develops the ability to think about his structures and becomes more analytical.

In the construction stage abstraction may play a crucial part. The realization that "this is just the same structure as that one over there, although they appear at first sight to be different," i.e. an insight into an isomorphism is often a necessary step before a really new, original construction can be made, according to Dienes (1964). Having constructed the new mathematical structure, the logical or analytical work must follow, of which mathematical generalization is but one example.

Dienes classifies generalizations as explicit or implicit. If the generalization is explicit, that is to say, if there is an awareness that a certain rule structure extends to a wider set of instances than was previously thought, then it can be symbolized in the general form. However, he maintains that there must be two-way traffic between a generalized situation and its symbolization. The symbolization will not be fully operative unless it can be interpreted. Hence the child must be able to give instances of actual situations that are symbolized by the symbols. These views of Dienes on symbolization and interpretation have important implications for ideas on evaluation of the child's mathematical learning.

Children performing at the concrete operations level are very limited in their ability to perform mathematical generalizations according to Dienes. He cites the generalization from a limited class of numbers to that of any number as an example of a generalization that takes a long time to become explicit for a child. With regard to the symbolization of explicit generalizations or abstract concepts Dienes states that, just as the research mathematician often invents his own symbols, when a child is truly ready to symbolize, not only will he be capable of doing so, but he will ask for appropriate symbols, and that perhaps this is the most desirable situation. Dienes (1963) has formulated a psychological principle governing this notion, but its uncertainty, in his view, is brought out by the following statement:

> There is also the question of whether symbolism can be used as a tool for cutting through irrelevant noise during the abstraction process, or whether it can only be used to formulate what has already been abstracted. (pp. 160-61).

There are several other psychological principles that Dienes has formulated with regard to mathematics learning. The principle concerned with symbolization stated above is known as the Principle of Dynamic Symbolization. The Principle of Multiple Embodiment refers to the use of embodiments of a structure as an aid to abstraction. When the rules for manipulating elements of embodiments are amenable to discovery and isomorphic to the properties of a mathematical structure, Dienes feels that the learning of the structure is facilitated by experiences with a variety of the embodiments. However, he believes that several different kinds of embodiments are necessary lest the concepts become tied to the one type and are never adequately generalized as a result.

To encourage generalization, the values of the mathematical variables involved in the concept must be allowed to vary. This Dienes calls the Principle of Mathematical Variability. Preliminary, structured and practice games must be provided as necessary experiences from which mathematical concepts can eventually be built. This is called the Dynamic Principle. It is easy to see the origin of this principle in his ideas on the cyclic process of concept formation and his notions of mathematical structure. Likewise, the Constructivity Principle, in which Dienes indicates the importance of constructive activity preceding analysis, can easily be traced to the concept formation cycle idea.

Finally, according to Dienes' Physical Principle, visual, tactile, and muscular imagery must accompany the learning of abstract mathematical concepts in the initial construction stage. This principle is of extreme importance since it does not necessarily apply beyond the concrete operations stage of development in Piaget's theory. However, with this principle Dienes is emphasizing the importance of pure perceptual thought for all concept formation, regardless of developmental level of the learner.

Piaget's theories form the basis of Skemp's (1963) ideas on how concepts are learned. He calls this the Schematic Theory of Learning. Mathematical concepts consist mainly of class, relational, and operational types. New concepts are assimilated to or accommodated by existing conceptual knowledge largely by means of the process of abstraction. They may be subsequently extended by the process of generalization.

Although concepts do arise from sensori-motor experiences, the development of new mathematics is almost entirely conceptual. In other words, primary concepts constitute only a small part of the totality of mathematical ideas. According to Skemp this notion is at variance with

Dienes' views.

> We are I think in full agreement that to enable the pupil to form a new concept we must give him (or her) a number of examples from which to form the concept in his own mind. For this purpose Dienes had produced some ingenious and attractive concrete embodiments of algebraic concepts - balances, peg-boards, colored shapes and frames, and the like. But these, I think, fail to take into account the essential difference between primary and secondary concepts, that only primary concepts can be exemplified in physical objects, secondary concepts can only be symbolized. (p. 44).

The essence of this statement is that even though both types of concepts are formed by abstraction, only primary concepts can be learned by manipulating physical objects, secondary concepts can only be learned in symbolic contexts.

For Skemp all mathematical concepts are either primary or have their roots in primary ones. Primary concepts are learned from concrete sensory experience of objects in the world outside of the mind. These concepts are the subordinate bases of the hierarchical network of secondary concepts which constitutes the structure of mathematics.

Symbols

In this section Dienes' and Skemp's ideas on mathematics, thinking and learning are interpreted in relation to the role of symbols in the learning of structural relations of mathematics.

Mathematically speaking, Dienes (1965) attributes two inseparable meanings to the term symbolization. There is first of all an abstract aspect in which the symbolization represents a structural relation. Particular symbol meanings are in a sense irrelevant to structural meaning, since the structural relations represent the common essence of a number of different situations. However, this is a rather hollow interpretation according to Dienes since mathematics is inevitably about the real world. Hence the second aspect of symbolization, the concrete one, assumes equal status. In this case the symbolization of the structural relation represents some experiential content to which it applies. This content is derived from the individual concrete meanings ascribed to the symbols. By way of contrast, Skemp believes there are two different types of symbols rather than two aspects of symbolization. Primary symbols refer to primary concepts and have only a concrete interpretation. Secondary symbols represent secondary concepts. Symbolizations which make use of secondary concepts are statements of structural relationships but do not have a direct concrete interpretation. They must be interpreted in terms of other sets of symbols which may be traced back to primary concepts having concrete content.

Concerning the role of symbols in thinking, the two fundamental differences between Dienes' and Skemp's ideas are in the type of concept involved, and the level of cognitive growth of the thinker. Areas of agreement include the notion that symbols are thought of as a means of fixing structures in the mind. Also, they can serve the role of additional embodiments of a structure and may be used to generate new structures. Of course, Skemp places more emphasis on the latter use because it involves analytic processes needed in learning secondary concepts.

On the one hand, as far as learning is concerned, at least for concrete operational children, symbols play little or no role in Dienes' view. The learning of structures is accomplished by constructive activity using physical objects or mental images of these. Symbols may be used in analyzing structures in order to generate new structures, but at the risk of impeding learning if this is carried on too long without reference to the concrete interpretation of the symbols. Hence, though symbols may enhance or impede the learning of structural relations, they are not central to the learning. The mechanisms of fitting symbols into learning so that it is enhanced, that is to say, the form of symbolic embodiment, and the order of introducing symbolic and concrete embodiments of a structure are open questions to be determined empirically, according to Dienes. On the other hand, Skemp feels that symbols play a key role in learning mathematics. Since such learning involves mainly secondary concepts which can only be symbolized, the bulk of learning takes place in symbolic contexts. Concepts are built up in hierarchical fashion, primary ones as abstractions and generalizations of physical experiences, and secondary ones as abstractions and generalizations of symbols which serve as mental objects. The Skempian view of the role of symbols in the learning of structural relationships of mathematics might be thought of as a semi-concrete interpretation as compared to the concrete interpretation of Dienes' way of thinking. The symbols at each stage, being arranged in a hierarchy of progressively higher level concepts with concrete experiences at the bottom, take on a concrete existence in that they are meaningful in terms of all the lower levels of the hierarchy subsumed.

Related Research

Dienes' research (1963) has been mainly of the descriptive type. He has provided for children, mathematical experiences with concrete materials. The rules governing manipulation of these materials are embodiments of the mathematical structures. He observed a wide range in the children's abilities to master the underlying concepts. He tested his psychological principles governing mathematics learning and found that adherence to them did facilitate the formation of the concepts in so far as most children were able to demonstrate their comprehension by manipulating the material in appropriate ways, but there was a sex difference in the children's performances in favor of boys. He attempted to introduce symbols at various stages in the concept development and found that it usually inhibited the learning of the concepts if the symbols were introduced prior to the completion of the analysis stage. He also found that children could symbolize and interpret best when they reached the stage where they asked to be given a way of symbolizing the concepts.

In a later, more formal kind of experiment, designed to determine the strategies of thinking in discovering the properties of various group structures, Dienes (1965) used a purely symbolic type of treatment. He found that some subjects used operational or pattern strategies while others relied on memory. The operation strategy is predicated on the assumption that any given symbol acts on each of the other symbols in a consistent way in producing a resultant symbol. The pattern strategy involves the assumption that patterns in the combinations of the symbols exist, and that discovery of these patterns will facilitate prediction of results. The operational strategy is the one most likely to aid the subjects in discovering the actual mathematical characteristics of the group structure, whereas the pattern strategy would more likely lead to the discovery of such things as symmetry. Those who used the operational or pattern strategies learned the concepts involved better, and were able to transfer this knowledge to isomorphic structures more successfully than the memory subjects. The operational strategy proved to be better than the pattern strategy for the more complex mathematical groups. As a result of this research Dienes suggested the need for an experiment involving symbolic versus experiential-cum-symbolic treatments. His main concern regarding the purely symbolic treatment seems to be that lack of constructive experience with concrete objects embodying the structure will lead to the formation of concepts that are not fully operational. That is to say, the child will not be able to apply the concepts to real situations.

Two of Skemp's investigations (1958) are relevant to this study. The first was a card sorting experiment in which the subjects had to determine which combinations of the concepts of size, color, and shape were being used. He found that those who were consciously aware of the concepts used, formed and transferred the concepts better than those who were not aware. In the second investigation Skemp found reflective intelligence, as measured by a test of his own design, to be significantly correlated with mathematics achievement. He used these results in support of his theory that reflective intelligence is necessary for success in mathematics.

Scandura (1967) conducted an experiment involving two treatment factors. The first factor was the form in which mathematical rules were given to subjects, either succinctly worded English, or initially unfamiliar mathematical symbolism. The second factor was the presence or absence of symbol pretraining. For instance, the Greek rule in the initially unfamiliar mathematical symbolism was $\sum_{z=1}^{t} z \left(\sum_{x=1}^{r} x + \sum_{y=1}^{s} y \right)$. The English version of the Greek rule was: 1) Add the consecutive integers 1 through r. 2) Add the consecutive integers 1 through s. 3) Add the sums obtained in 1) and 2). 4) Add the consecutive integers

1 through 5. 5) Multiply the sums obtained in 3) and 4). The symbol pre-training for this rule was: to find the number equal to $\sum_{r=1}^{k} r$ add the integers from 1 to k. For instance, $\sum_{t=1}^{6} t = 1 + 2 + 3 + 4 + 5 + 6 = 21$. The subjects in the pre-training treatment groups were then given several similar examples to practice. One criterion measure was problems involving the rules given to the subjects. For each test item the subjects were given a problem involving a particular rule and given the name of the rule to use in solving the problem. A second criterion was the time taken to learn the rules. On the first criterion he found no difference between the English groups and the pre-trained symbol group on applying the rules to problems. There were no cases of correct application for the symbol group which did not undergo symbol pre-training. On the second criterion measure, speed of learning, he found that symbol pre-training had no effect on the English group, but that the pre-trained symbol group learned more rapidly than the symbol group that was not pre-trained. Both symbol groups learned more rapidly than both English groups.

The results of Scandura's research seem to suggest that the symbolic mode of presentation preceded by an explanation of symbol meanings is the best method because students can learn to memorize the rules more rapidly and apply them as well as they can apply the English statements. However, the expository method which encouraged rote learning was used in the presentation of the rules in symbolic form.

Perhaps the presentation of the symbolic statements in a structured context whereby students are encouraged to discover the rules for themselves might lead to different results. The next investigation seems to indicate that this is indeed the case.

In a series of experiments designed to teach the structure of groups, Avital (1966) used treatments based on age distinction between first and second order symbols. In terms of Skemp's notion of primary and secondary concepts, second order symbols would refer to secondary concepts. Avital says that "mathematical symbols are mostly second order symbols, that is, symbols introduced by and explained with the help of first order symbols, and replacing them." (p. 3).

In the first experiment he used some formulas from basic statistics presented in their usual symbolic form for the second order symbol treatment, and statements of the same formulas in ordinary English for the first order symbol treatment. The subjects were vocational school teachers. The experiment was not tightly controlled and the results were inconclusive, but showed a trend in favor of the ordinary English treatment. However, the criterion test was strictly in terms of ordinary English.

In two subsequent experiments conducted by Avital, in which the content taught was the structure of the mathematical group, three treatments were used. The mathematical symbolic treatment emphasized the operational group characteristics presented in the form of second order symbolic statements. The pattern treatment emphasized the symmetry of the pattern when the products of all pairs of elements were arranged in an array and presented as second order symbolic statements. In the verbal descriptive treatment the functioning of each element was described in words from the point of view of group structure. There were two groups of subjects, student teachers and high school students. The pattern treatment proved to be the most effective one for both groups of subjects taken together while the mathematical symbolic was the second best for the high school students, and the verbal was the best for the student teachers. Avital felt that the superiority of the pattern method was due to its symmetry. He also felt that the reason the teachers succeeded better in the two experiments with verbal description rather than second-order symbols was because they were not so recently involved in working with mathematical symbols as the high school students.

In the third experiment Avital used the same three treatments with only high school subjects, but changed the content to a less symmetrical structure, the cyclic group. In this case the mathematical symbolic treatment proved to be the most effective. Another important finding was that there was no significant difference in performance between boys and girls. He concluded that this points to the fact that sex differences in mathematics achievement are mainly culturally induced. Significantly, the criterion test included backward association and transfer items as well as recall. This experiment, which included students from grade nine, ten and eleven in Ontario revealed a significant grade effect in favor of the higher grades. Avital said that this result supports the finding of the previous study that continuous involvement in learning mathematics transfers to the learning of new structures.

Behr (1967) conducted an experiment in order to search for evidence that might suggest the influence of mental factors on school learning situations. He also hoped to find out whether it is possible to design instructional materials in a way that would suit the learner's mental ability profile. Using Guilford's theory of the Structure-of-Intellect and Gagne's theory of Task Analysis, Behr designed two programmed presentations of the concepts of modular seven arithmetic. The figural-symbolic presentation employed a figural model and diagrams for the presentation. The verbal-symbolic-program presented the same concepts verbally. Behr found significant interactions between five of fourteen selected factors of mental ability and the two presentations. The trends in the results indicated that persons high in figural, or verbal or symbolic aptitudes were able to learn better if the materials were presented in these respective forms. However, Behr conceives of the results of his investigation only as sufficient encouragement for further research.

Dienes' research has produced important findings. Unfortunately, few experiments of the sixties were closely enough related to these findings to contribute to or clarify Dienes' theory. The studies cited here do, however, point to the fact that there is a need to investigate the role of symbols in mathematics learning. Dienes' theoretical stance and his experimental findings, together with the related studies reported here, serve as the basis of the design and development of this study as described in the next two chapters.

In the seventies, subsequent to the design of this study, several other related investigations were carried out. One such experiment was a follow-up study by Dienes and Jeeves (1970) of the earlier study of structural thinking. In this follow-up the focus of attention was on transfer of structural relations. Mathematical group structures served as the content to be learned and transfered. The most important finding of this study with respect to the role of symbols was that perceptual cues helped in the identification and learning of the structural relations. Starting with a nine element group, the subjects were required to transfer their knowledge of its structure to its various three-element sub-groups. Those sub-groups for which a perceptual cue, such as invariant color or shape, was provided were learned more readily than those for which no perceptual cue was provided.

Several other studies closely related to ideas of Dienes have been conducted. The most directly related study is that reported by Branca and Kilpatrick (1972). The investigators were attempting to validate the findings of Dienes' (1965) study of structural thinking in which the operator, pattern, and memory strategies were studied. The pattern of subject evaluations of strategies used, and consistency of strategies across tasks, as reported by Dienes, were confirmed in the Branca and Kilpatrick study. However, Dienes' finding of a relationship between evaluations and strategy scores was not confirmed. This latter result led the authors to conclude that the strategies postulated by Dienes have yet to be demonstrated convincingly.

Several other studies indicate the inconclusiveness of experiments related to Dienes' theoretical ideas. Suydam and Higgins (1975) in a review of research on activity based approaches to mathematics teaching reported that three of four studies found no significant differences in achievement when number of embodiments was varied. In studies comparing student mathematics achievement under concrete, pictorial and symbolic learning sequences, the most successful approach was a combination of pictorial and concrete manipulative materials, with only three of twenty-eight studies reporting superiority for symbolic sequences alone.

One study supporting the conclusions of Suydam and Higgins was that conducted by Austin (1974). The three methods compared were based on Bruner's three levels of representation of thought. The manipulative-pictorial method was derived from the enactive level, the pictorial method was allied to the iconic level, and the symbolic method was based on the symbolic level. In a comparison of the mathematics achievement of students subjected to these methods, the manipulative-pictorial and the pictorial methods proved generally superior to the symbolic method.

A study representative of the three showing superiority for the symbolic method was conducted by Fennema (1972). In this study recall and transfer of a mathematical principle taught to second graders by two different methods was investigated. The methods compared were meaningful concrete, in which the principle was demonstrated and practiced using cuisenaire rods, and meaningful symbolic in which the principle was taught using numeral, operation and relation symbols only. The principle taught was multiplication as repeated addition. The results showed no significant differences in recall, but significant differences in both concrete and symbolic transfer favoring the symbolic treatment group were found.

These studies of the seventies indicate increased interest in both Dienes' ideas and the role of symbols in mathematics learning. The Branca and Kilpatrick study indicates that Dienes' notion of strategies of thought is far from well established as yet. The review of Suydam and Higgins indicates that other facets of Dienes' theories also require more experimental support, particularly his ideas on multiple embodiments and the role of symbols. The conflicting results obtained point to the need for continued imaginative research on these topics.

CHAPTER III

DESIGN OF THE STUDY

The Logic of the Study

Three Alternative Approaches to the Role of Symbols

One of the most attractive features of Dienes' theory is that it deals not only with the role of symbols but also with the broader problem of how mathematical structural relations are learned. However, Dienes advances only tentative propositions on how best to facilitate such learning. His theory has been and remains evolutionary. In addition, some points are vague, so inferences are tenuous. Some significant aspects are not addressed, gaps exist, and many questions are raised but not answered. Nevertheless, in spite of its aura of uncertainty, Dienes' theory is the only one that comes close to grappling with the problems under investigation in this study. On this basis alone Dienes' theory seems eminently qualified to serve as the framework of an instructional approach specifying a particular role for symbols in the learning of structural relations of mathematics.

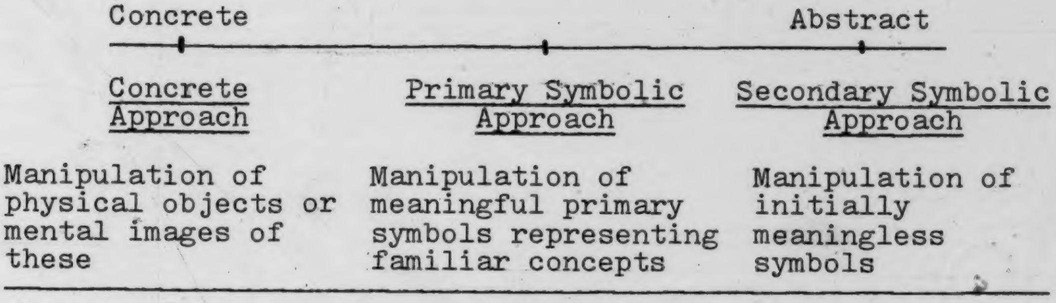
In view of Skemp's criticism of Dienes' views, it seems reasonable that formulating two approaches, one based on Dienes' theory, and another a modification of this to take Skemp's arguments into account, could serve as the basis of an investigation. The two such approaches could then be compared to each other with respect to their effects on the learning of structural relations in mathematics. In order to determine whether any other alternatives should be considered, the two discussed above must be set into a theoretical framework. The most obvious framework which comes to mind is the concrete-abstract dimension. In this dimension the concrete end is the most tangible and is occupied by real physical objects and events. The abstract end is intangible, it is the realm of ideas far removed from material referents. The middle ground is occupied by concepts somewhat removed from material referents yet tangible to some extent in that they are familiar and meaningful.

The Dienes approach, henceforth referred to as the Concrete approach, makes use of the manipulation of concrete materials in the learning of structural relations. Symbols are relegated to the role of labels to be attached some time after initial abstraction, though once attached they may be used whenever the learner engages in analysis. Hence, this approach can be thought of as occupying a position near the concrete end of the concrete-abstract dimension.

The modification of Dienes' approach to satisfy Skemp's criticism, from now on referred to as the Primary Symbolic approach, can be thought of as occupying a middle ground on the concrete-abstract dimension. In this approach the structural relations of mathematics are considered to be secondary concepts. Their learning is based on the utilization of other more basic concepts known to the learner. These other concepts are represented by primary symbols which are manipulated in order to render the learning of the structural relations. Though such an approach might be considered closer to the abstract by virtue of its primary symbol context, it can be considered to approach the concrete from the point of view of the tangibility of the familiar and meaningful conceptual basis of the learning. Hence, the Primary Symbolic approach rightfully occupies the middle ground along the concrete-abstract dimension.

We could conceive of still another approach, a truly more abstract symbolic approach, which would consist of learning the structural relations through manipulation of symbols devoid of initial meaning. It is through the manipulation of the symbols that they take on meaning and lead to the learning of the structural relations. This approach, which will be called the Secondary Symbolic approach, represents the abstract end of the concrete-abstract dimension. The relationship of the three approaches to the concrete-abstract dimension is depicted in the diagram of Figure 1.

Though the Secondary Symbolic approach appears to be a logical third alternative for the learning of structural relations in mathematics, its inclusion as an instructional approach is supported by other considerations. The major



Concrete Approach	Primary Symbolic Approach	Secondary Symbolic Approach
Manipulation of physical objects or mental images of these	Manipulation of meaningful primary symbols representing familiar concepts	Manipulation of initially meaningless symbols

FIGURE 1

The Three Approaches on a Concrete Abstract Dimension

advantage of such an approach is that it offers a symbolic alternative which is not directly dependent on arithmetic achievement. Whereas the Primary Symbolic approach would draw on the students' knowledge of arithmetic for much of the structural relations learning, the Secondary Symbolic approach does not draw on such knowledge. Though more abstract and less meaningful initially, the Secondary Symbolic approach may be viewed as simpler in some ways than either of the other two approaches. If in the Secondary Symbolic approach symbols are viewed initially as meaningless, they will carry no irrelevancies, misconceptions and complexities that might be associated with more meaningful approaches. Learning should be focussed directly on the structural relations rather than the myriad of associations and connotations possibly carried along with the other two approaches.

A major task of this study is to elaborate on these three approaches which all offer alternative roles for symbols in the learning of structural relations of mathematics. Dienes' theory serves as the basis for all three approaches. One approach represents Dienes' ideas as closely as possible. The other two approaches are modifications. These modifications affect only the role of symbols and are motivated by the possibility of flaws in Dienes' principles that concern symbols.

The Concrete Approach. This approach is based on principles derived by Dienes from certain ideas of Bartlett, Bruner and Piaget, as well as results of his own research. The unique aspect of the approach is the Physical and Dynamic Symbolization Principles. The Physical Principle originated as a hypothesis written by Dienes (1959). He subsequently dropped the idea as a separately identified principle, but it remains implicitly embedded in his theory. The demise seems to be related to Dienes' development of the notion of manipulation of mental images through stories. Therefore, for the purpose of this study the Physical Principle will be interpreted as follows: manipulation of physical objects is necessary in the constructive stage of the learning of mathematical structural relations by children. As children grow older, this can be replaced increasingly, but never fully, by manipulation of mental images in the context of story embodiments of mathematical structures.

The principle dealing most directly with the role of symbols is Dynamic Symbolization. Generally speaking, this principle requires that symbols not be introduced until after the structural relation has been abstracted by the learner. Whenever possible children should be allowed to create their own symbols. If it is necessary to provide symbolization for the learners, the symbols should take on forms that progress through iconic to conventional representation.

Both principles are derived from Dienes' interpretation of Piaget's and Bruner's stages of development of thought and representation. The notion that thought of concrete operations children must be accompanied by manipulation of real objects or mental images is one such derivative idea. Another is Bruner's suggestion that this is also necessary in the formation of any new concept regardless of the age of the learner.

The Primary Symbolic Approach. This approach offers an alternative to Dienes' Physical and Dynamic Symbolization Principles as a result of their criticism by Skemp. Though much of the approach adheres to Dienes' principles, those concerned with the role of symbols are rejected and replaced by principles in line with Skemp's views. The basis of these principles is Skemp's belief that most of mathematics is comprised of secondary concepts that can be represented only by symbols, and thus must be learned in the context of symbols. The symbol context represents concepts known to the learner, on which the secondary concepts to be learned are based. To follow Dienes' approach of formulating principles for optimal learning, one principle based on these ideas of Skemp, referred to henceforth as the Schematic Principle, requires that a structural relation must be learned in the context of symbolic representations of prerequisite concepts.

The Secondary Symbolic Approach. This approach was suggested as a logical third alternative occupying the abstract end of the concrete-abstract dimension. Under such an approach the learning of mathematical structural relations takes place in the context of meaningless symbols. In the realm of mathematical philosophy such an approach would be classified as formalistic. Bruner appears to cast favor on this approach in his introduction to Dienes (1963).

> . . . Dr. Dienes is much more distrustful of 'formalism' than some of the rest of us. The symbols of a language - a natural or a mathematical language can either be viewed as transparent or opaque. When we treat symbols as transparent we are principally mindful of the referential function, what they 'stand for' or 'mean'. But it is also possible to treat a symbol system without regard to what lies beyond the symbols in the world of experience, to treat the system as a self sufficient body of rules for forming and transforming sentences or equations or functions . . . . Many mathematicians today believe that one can indeed introduce many ideas in mathematics by first giving children a sense of the algebraic structure and then moving on to the question of embodiment or reference. (pp. xi-xii).

Dienes does recognize the possibility of productive learning emanating from the formalist approach as the following quote testifies: "There is the question of whether symbolism can be used as a tool for cutting through irrelevant noise during the abstraction process, or whether it can only be used to formulate what has already been abstracted." (pp. 160-161). Obviously he is not dogmatic about the usefulness of a formalistic approach. In the introduction to Dienes (1966), Sandford and Williams raise the question of whether concrete materials sometimes hinder learning by distracting the learner from essentials. This further supports the idea of formulating a formalistic approach free from extraneous distractions. The Secondary Symbolic approach calls for rejection of the Physical and Dynamic Symbolization Principles, and substitution of these by a principle expressing the essence of formalistic learning. This principle, referred to as the Secondary Symbolic Principle, calls for learning of structural relations of mathematics by means of manipulating symbols which have no initial meaning attached to them.

Treatments

The review of literature for this study suggested alternative approaches to the role of symbols in the learning of structural relations of mathematics. On this basis the rationale for three such approaches was developed. In order to develop treatments representing these three approaches specific mathematical content had to be chosen. The content chosen was the structural relationships within Abelian groups. There were two main reasons for this choice. First, Dienes considers group structures as basic mathematics that can be learned by young children. He has conducted a great deal of research (1963 and 1965) into the teaching of this topic to children and adults. Second, the usual defining properties of a group, such as associativity, constitute structural relations considered important in the study of numbers. The actual treatments are direct applications of the three alternative approaches to the teaching of Abelian group properties. Thus the three treatments developed are the Concrete, Primary Symbolic and Secondary Symbolic treatments. The development of these treatments is described in the next chapter.

Criterion Variables

Since all three treatments are based on Dienes' theory, which is intimately tied to his beliefs on the nature of mathematics and mathematical thinking, the criterion variables are derived from an analysis of what Dienes considers to be the important outcomes of learning structural relations of mathematics.

Dienes' (1960) assigns equal status to the two faces of mathematics, pure and applied. Hence both pure mathematics and applications should be represented in the learning outcomes. The pure side is structural relations. The structural relations within the group include the properties or rules such as closure, existence of an identity element, and so on. Concerning structural relations between structures, recognition of isomorphisms and extension by generalization are of prime importance. These two are referred to by Dienes (1965) as transfer. In his view transfer refers to the learning of any given structure by the utilization of knowledge of a similar structure. The knowledge of similarity of structure is brought into play by the use of different kinds of relationships between structures. For group structures one such important relationship is recursion.

There are two kinds of recursion, generalization and particularization. Generalization is the extension of a structure to one which includes a wider range of elements but follows the same rules. Particularization is going from one structure to an isomorphic image of a subset of the structure which follows the same rules.

Dienes (1966) has defined the learning of mathematics as not only the comprehension of structural relations and the ability to apply these to the world, but also the symbolization of the relations. He believes that symbolization is more than just attaching labels to structures. The symbolization must be an abstraction. It must represent a structural relation which is known to the learner as the common essence of a number of different situations. In addition it is viewed as a capping off of the learning, making it complete by fixating the structure in the mind. But the symbolization is not fully operative, according to Dienes, unless it can be interpreted. That is, given a symbolic representation of a structural relation, a non-symbolic embodiment to which it applies can be cited.

In summary then there are four major outcomes of the learning of structural relations of mathematics in Dienes' view. These are symbolization, interpretation, transfer and application. The actual structural relations are of two types, those between structures and those within. Recursion is the main type of relationship among group structures. The defining properties of the group structure such as closure are the main types of structural relations within the group structure.

Learner Related Variables

According to Skemp (1960), it is only through analysis using reflective intelligence that one learns mathematics. If this is true then only those with sufficiently high reflective intelligence will be able to learn mathematical structural relations such as those associated with the algebraic group structure, regardless of treatment used. Dienes (1964) agrees with the necessity of using reflective intelligence, but only during the analytic stage. He disagrees with Skemp on when analysis is necessary for learning mathematics. He believes analyzing is observable in young children, although it reaches full maturity during the teenage years. But he feels that much mathematics can be learned by young children through constructive activity. Thus reflective intelligence should be examined experimentally to ascertain whether it is a factor in the learning of structural relations of mathematics.

Similarly, arithmetic achievement and sex should be examined for the same reasons. Dienes (1960) states that, " . . . one experience may have effects on the learning of concepts in several widely different branches of mathematics, and the effects may not be observed until a long time, perhaps years, after the experience." (p. 29). The effects of arithmetic achievement on the learning of structural relations associated with algebraic groups may very well be a case in point. Some of the structural relations within a group, such as commutativity, often are used and intuitively understood by children when they deal with numbers and operations of arithmetic. Thus how well they achieve in handling operations on integers and fractions may well be related to how well they are able to abstract the structural relations of the group. There might even be some interaction among treatments and levels of arithmetic achievement since arithmetic embodiments are very likely to be used as a primary symbol context. Finally, sex should be examined, for although research findings on sex differences have been inconclusive, Dienes (1960) himself observed sex differences in learning structural relations by subjects in his experiments.

The Experimental Study

Experimental Questions

The questions to be examined in this study are as follows:

1. Among the three treatments, Concrete, Primary Symbolic and Secondary Symbolic are there differences in achievement in symbolization, interpretation, transfer and application of the relations of the group structure?

2. Are there differences in achievement on symbolization interpretation, transfer and application of the relations of the group structure between

   a) high and low reflective intelligence children,
   b) high and low arithmetic achievement children, and
   c) boys and girls?

✓

3. Is there any interaction between treatments and
    a) reflective intelligence,
    b) arithmetic achievement, and
    c) sex?

Experimental Procedures

The experiment took place in the Spring of 1969 and was preceded by two pilot studies in 1968. The pilot studies were used to refine the treatments and tests, and resulted in the decision to conduct the experiment at the grade seven level. At the outset subjects were pre-tested on reflective intelligence and arithmetic achievement. Following this the subjects underwent the treatments for a period of four weeks. This amounted to a total of twelve forty minute class periods for the treatments. Much of the instructional material was printed on cards available for individual subjects or groups of three or four. However, general directions and individual guidance were given orally by one or other of the investigator and one other instructor. Immediately following the treatments the criterion tests of symbolization, interpretation, transfer and application, known as Group Concepts tests, were administered.

The Sample

The sample consisted of 168 grade seven students of one junior high school in Edmonton. These students were randomly pre-assigned by sex to six classes of equal size. Each class had an equal number of boys and girls. The classes were then randomly assigned in pairs to treatment groups.

Permission to publish I.Q. and achievement data was not obtained, but comparison of the sample to the population of the school system indicated comparability.

Learner Related Measures

The following reflective intelligence measures were used: Skemp's test of Concept Formation (SK 4)(1) requires subjects to identify whether each of three test examples are exemplars or non-exemplars of a certain geometric concept after having been given three exemplars and three non-exemplars.

Skemp's Test of Reflective Action with Concepts (SK 4)(2) is designed to be administered immediately following (SK 4)(1). It involves logical multiplication: combining two concepts to form a new concept in which possession of both of the properties of the original concepts is used as the criterion. Three exemplars of the double property and three non-exemplars differing from each other are given. The subject is required to indicate which of several test examples are exemplars and which are non-exemplars. Skemp calculated a reliability coefficient of 0.76 for this test.

Skemp's Test of Operations Formation (SK 6)(1) requires the subject to operate on test figures using operations illustrated on a demonstration sheet by means of three examples. Skemp obtained a 0.94 reliability coefficient for this test.

Skemp's test of Reflective Action with Operations

(SK 6)(2) requires subjects to demonstrate combining and reversing operations on test figures. Skemp's reliability coefficient for this test is 0.95.

The arithmetic achievement test consisted of twenty computational items and word problems involving basic operations on real numbers. These items were taken directly from the Van Engen et al (1963) text used by the subjects in regular grade seven mathematics program. Samples of the final forms of the learner related measures are presented in Appendix B.

Experimental Design

The properties of the two mathematical groups of order four were taught to subjects by means of one or other of three treatments. These properties such as commutivity, existence of inverse elements, and so on constituted the structural relations within the group structure to be abstracted by inductive processes. As far as possible the three treatments differed from each other only in the role assigned to symbols. Dienes' other principles were emphasized in all three treatments in the initial phase, referred to as phase I. Eleven of the twelve class periods were devoted to this phase. In phase II of each treatment one embodiment from each of the other treatments were presented in an expository way, and practice exercises were provided. This phase utilized only the last of the twelve class periods.

In phase I of the Concrete treatment physical objects and mental images of real objects and events provided the context from which the properties of the group were to be abstracted. In phase II symbolic embodiments were given to provide enough familiarity with these to be able to use them in criterion test items. In phase I of the Primary Symbolic treatment familiar symbolizations of previously taught concepts served as the context from which abstraction of the group properties was expected. Phase II of this treatment consisted of the expository presentation of a meaningless symbol and a physical object representation of the group properties, followed by practice in using the properties in these contexts. For the phase I Secondary Symbolic treatment meaningless symbols were used as the learning context, and Primary Symbolic and Concrete embodiments were added on in phase II, along with practice in using the properties in these contexts.

The three 2 x 3 factorial designs used to test treatment and learner related effects and interaction were as follows: (1) The two categories of reflective intelligence, high and low versus the three treatments, Concrete, Primary Symbolic, and Secondary Symbolic. (2) The two categories of arithmetic achievement, high and low versus the three treatment categories. (3) The two categories of sex, boy and girl versus the three treatments. A schematic representation of this design is shown in Figure 2.

The learner related variable categories of high and low for reflective intelligence and arithmetic achievement

Learner Related Variable		Concrete	Primary Symbolic	Secondary Symbolic
(1) Reflective Intelligence	High			
	Low			
(2) Arithmetic Achievement	High			
	Low			
(3) Sex	Boy			
	Girl			

FIGURE 2

The Three 2 x 3 Factorial Designs of Learner Related Variables vs. Treatments

were established by assigning those above the median to the high category and those below to the low category.

In order to determine whether each of the four criterion Group Concepts tests could be statistically analyzed separately, multiple correlations on all possible pairs were carried out. The choice of the multivariate or univariate analyses depends on whether these tests are dependent or independent, so the results of the correlation study were used in determining this decision. In this case the analysis appropriate for the 2 x 3 designs was multivariate analysis of variance. These analyses of variance were used to test the null hypotheses with the group concepts tests scores used as the criterion measures.

Null Hypotheses

For grade seven students there is no difference at the five percent level of significance:

H_1 among treatment Group Concepts mean vectors,

H_2 between reflective intelligence category Group Concepts mean vectors,

H_3 between arithmetic achievement category Group Concepts mean vectors,

H_4 between sex category Group Concepts mean vectors,

For grade seven students there is no interaction at the five percent level of significance.

H_5 between reflective intelligence categories and treatments,

H_6 between arithmetic achievement categories and treatments,

H_7 between sex categories and treatments.

CHAPTER IV

THE DEVELOPMENT OF THE TREATMENTS AND CRITERION MEASURES

Introduction

This study consists of two main undertakings. The first is a theoretical development of approaches leading to the development of treatments and criterion tests. The second is a classroom experiment to test these treatments. The first undertaking consists of two parts. In the first part Dienes' theory is analyzed, and three approaches to the learning of structural relations of mathematics are developed. These three approaches, though based on Dienes' theory, differ in the role assigned to symbols. Dienes' theory also serves as the basis for deciding the mathematical content of the criterion tests. The second part of this task consists of using this analysis of Dienes' theory to develop three treatments and four criterion subtests to be used in the classroom experiment. The design of the experiment was described in the previous chapter. This chapter consists of a description of the development of the treatments and criterion measures. An attempt is made to show that these treatments and measures are valid representations of the principles on which they are based, and do represent what Dienes considers as structural learning of mathematics. Examples of each treatment and several test items are presented in order to enhance the description and support the validation. The final versions of all four criterion measures are included in Appendix B.

The Treatments

The mathematical content of the treatments is the two Abelian groups of order four known as the Klein and cyclic groups. The difference between these two groups is that each element of the Klein group is its own inverse, but for the cyclic group of order four only the identity and one other element act as their own inverses.

There are two phases to each treatment. The first phase is the one from which the structure should be learned by the subjects. The second phase is an expository exposure to one embodiment of each of the other two treatments to provide enough familiarity with these so that they may be used in the criteria tests.

Phase I of each treatment consists of four embodiments, two Klein and two cyclic four groups. Game or story forms of each of these were developed to comply with the basic principles governing all treatments, the Dynamic, Constructivity, Mathematical Variability, Multiple Embodiment, and Contrast Principles. In addition, each treatment was based on its own unique Principles.

The Concrete Treatment

The unique aspects of this treatment are based on the Physical and Dynamic Symbolization Principles. The examples of concrete treatment embodiments described below are intended to illustrate how they were developed in compliance with the basic and unique principles on which they are founded.

The Klein group has a physical interpretation known as the Symmetries of a Rectangle. The elements of the Group are movements, or motions as they are usually called, about points or lines of symmetry of a rectangle. A rectangular card, as shown in figure 3, was used as the physical object.

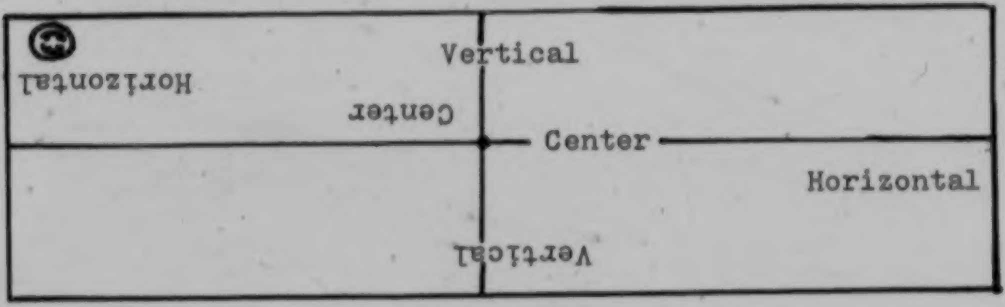


FIGURE 3

The Happy Face Card

A description of the four permitted motions was provided on another card. These consist of 180° turns about the vertical and horizontal axes and 180° and 360° rotations about the center point in the plane of the card. The motions are transformations which change the initial state of the card to any one of four possible states. Both sides of the card are considered to be equivalent, that is, any state is judged solely by the position of the vertices in the plane of the card. The four motions are elements of a set and any combinations of them performed one after another is an operation. Single motions or combinations are equivalent if they result in the same state. The structure of the rules for the operations is isomorphic to the Klein group.

A flip about the horizontal axis, H, takes the top left hand vertex, the 'happy face', to the bottom left. If this is followed by a flip about the vertical axis, V, the 'happy face' goes to the bottom right hand corner. Likewise, V followed by H produces the same state. For any sequence of moves the starting point is the state for which the 'happy face' is at the top left. Thus the situation described above may be represented symbolically as $H.V = V.H$, where the dot '.' means 'combined with' or 'followed by'. This is an instance of the commutative property of the Klein group.

The Dynamic Principle is one on which all treatments are based. According to this principle the subjects are required to engage in preliminary free play, structured and practice games, in that order. The word 'game', as used by Dienes, is interpreted broadly to mean activity engaged in by children, whereby decisions or choices by them are required.

For the free play stage the children were given a card like the one shown in figure 3. They were directed to lay the card on the desk top in front of them with the horizontal axis parallel to the front edge of the desk. In addition, the 'happy face' had to be at the top left hand corner. They were then asked to move the happy face to each of the other corners in as many different ways as they could. The subjects were seated in groups of three or four. They were allowed to discuss the activity freely within their groups and with an instructor.

For the structured games the students were given a description of the restricted set of moves horizontal flip, vertical flip, half turn (180° rotation) and whole turn (360° rotation). Subsequently, each group was given activity cards on which a series of questions about the restricted set of motions were posed.

For the practice games the more conventional type of game with rules and players was used. Cooperative rather than competitive play was encouraged by awarding points earned by individual players to the entire group. The accumulation of a sufficiently high group score allowed the group to move on to another game if they wanted to. Examples of these games are described later.

The Constructivity Principle was used in a number of different ways. First, in the free play the subjects were permitted to freely manipulate the card in order to build up events from which the group concepts could be abstracted. Discussion was allowed but not mandatory, so that at no time was there any specific request for the students to verbalize discoveries or to analyze situations.

As with the free play the initial emphasis in the structured games was on doing. However, certain restrictions were introduced to create rule bound play. The moves were restricted to the four permitted in the Symmetries of a Rectangle embodiment of the Klein group. The questions asked were designed to lead the subjects to construction of exemplars of the properties of the group structure. The subjects were asked to record some answers in tabular form so that this would be available for use later. The actual wording of the first activity card in the series is as follows.

Rectangle Game

1. Do the horizontal flip. What additional move can you make that would leave the 'happy face' where it now is?
2. Do the vertical flip. What other move done now would leave the 'happy face' where it is?
3. What move following the half turn leaves the 'happy face' in the same place as doing the half turn by itself?
4. Do a whole turn move. What happens to the 'happy face' if you now do another whole turn?

This card is designed to allow the subjects to create exemplars from which the identity element property might be abstracted.

Initially the data is recorded as shown by figure 4 in chart 1, Combinations of Pairs. The underlined moves are to be filled in by the subjects after the appropriate moves have been made. The chart is designed to emphasize patterns from which the properties of the group structure might be discerned.

After chart 1 was completed the subjects were shown by

Horizontal then Vertical is Half Turn

Half Turn then Horizontal is Vertical

Vertical then Half Turn is Horizontal

Horizontal then Whole Turn is Horizontal

Half Turn then Whole Turn is Half Turn

Vertical then Whole Turn is Vertical

Whole Turn then Whole Turn is Whole Turn

Horizontal then Horizontal is Whole Turn

Vertical then Vertical is Whole Turn

Half Turn then Half Turn is Whole Turn

Vertical then Horizontal is Half Turn

Horizontal then Half Turn is Vertical

Half Turn then Vertical is Horizontal

Whole Turn then Horizontal is Horizontal

Whole Turn then Half Turn is Half Turn

Whole Turn then Vertical is Vertical

Chart 1, Combinations of Pairs

FIGURE 4

First Data Recording Chart

an instructor how to use it to construct chart 2, the Lattice Table shown in figure 5.

The purpose of introducing the charts at this time was to allow the subjects who were ready to analyze to have an opportunity to do so. However, in accord with the Constructivity Principle that construction must precede analysis, no direct reference to the patterns of the charts were made. They were presented merely as recording devices.

	Whole Turn	Horizontal	Half Turn	Vertical
Whole Turn	Whole Turn	Horizontal	Half Turn	Vertical
Horizontal	Horizontal	Whole Turn	Vertical	Half Turn
Half Turn	Half Turn	Vertical	Whole Turn	Horizontal
Vertical	Vertical	Half Turn	Horizontal	Whole Turn

Chart 2, Lattice Table

FIGURE 5

Second Data Recording Chart

Since the charts make use of all possible cases of pairs of elements the Principle of Mathematical Variability must be complied with in order for the charts to be completed.

The Principle of Contrast was also used at the structured game stage. One or more properties was set in contrast for each embodiment. For the Symmetries of a Rectangle embodiment the closure property was used. Prior to the restriction of the structured games to the eventual four moves a fifth move was also used. It was introduced as follows:

Diagonal Guide

1. Draw a line from the bottom left hand corner to the top right hand corner of your 'happy face' card.
2. Place the card on the desk in front of you in the normal starting position ('happy face' at top left.)

3. Begin to turn the card about the center as you would for a half or whole turn but stop when the line you drew is parallel to the edge of the desk in front of you. This is called the quarter turn move.

4. Can you make any pairs of other moves that would leave the 'happy face' in the same place as making just the quarter turn move?

5. What single move is the same as two quarter turn moves?

6. Is there any single move that puts the 'happy face' in the same place as three quarter turn moves?

Numbers 4 and 5 are designed to highlight the idea of closure, and number 6, being a non-exemplar of closure, places the concept in contrast. After this experience the quarter turn is abandoned as an aspect of the embodiment.

In the practice game stage a narrower and more usual interpretation of the term 'game' was used, but care was taken not to violate any of the basic principles as a result. The Constructivity Principle was maintained, but, analysis was provided for in the permissive sense described before, as it was increasingly possible for the subjects to study the properties through the patterns of the charts. The charts were used to verify the results of moves in the games, but, if in doubt, the actual moves could still be made. Each practice game was designed to highlight one or more of the group properties such as commutivity. The actual card describing a commutative game played by the subjects is as follows:

Rock and Roll

1. Use your 'happy face' card to play this game in

which only the following moves are allowed:

a) Horizontal Rock - flip about the horizontal line.
b) Vertical Rock - flip about the vertical line.
c) Whole Roll - a whole turn about the center.
d) Half Roll - a half turn about the center.

2. Take turns at being the dealer for each different game you play. The dealer starts each play by calling two moves.

3. Each player takes turns at answering. On his turn the player calls two moves he thinks will put the 'happy face' in the same place as the dealer's two moves.

4. Dealer checks the charts to see if player correct. (All disagreements should be appealed to an instructor). A game is completed when each player has had a turn.

5. Score:

   a) one point for correct answer if both player moves are different from either dealer move,
   b) two points if one move is the same,
   c) three points if both moves are the same but not in the same order.

6. You can pick out a new game card and start playing any time after your group has scored a total of 20 or more points at this game.

In order to ensure that construction would precede analysis the subjects were never required to analyze. However, through the structure of the games and the patterns by which results were recorded, they were provided with ample opportunity to analyze if they were so inclined.

In compliance with the Dynamic Symbolization Principle, symbols were withheld until they appeared to be needed. In all cases they were introduced just prior to the start of practice games as it was assumed that for practice to be meaningful it should be preceded by abstraction.

Appropriate symbolism was to be discussed and preferably chosen by the subjects, but if imposed it would be done in stages if necessary, starting from symbols endowed with much meaning of the referant to arbitrarily selected ones. Allowance was to be made to accelerate the process by even earlier introduction of symbols where a clear indication of desire by the subjects was in evidence. In the case of the Symmetries of the Rectangle embodiment it was felt that initials, H for horizontal, V for vertical, T for half turn, and W for whole turn might be suitable, but iconic representation such as ↑ for vertical, and ↻ for whole turns was also considered. The actual choice was to depend on the preference of the subjects, and needed to be standardized only within groups.

The Principle of Multiple Embodiment was followed by using three other embodiments. Two of these were of the game type, Tire Rotations and Color-Shape, and on the story type, Gymnastics. In accord with the idea of maximal perceptual variability these embodiments were chosen so that the objects were quite different in physical properties. The embodiments chosen were identical or very similar to ones described by Dienes (1967). Each of these embodiments was developed in the same way as the Symmetries of the Rectangle in order to comply with the principles on which the treatment is based.

Phase II of the treatments was intended to balance the outcomes for criterion comparison purposes, but was not intended to bring about the initial learning of the structures involved. For the Concrete treatment it is necessary to present one embodiment of each of the other two treatments as applications of the structures learned by the Concrete treatment subjects. This is necessary so that questions pertaining to these embodiments may be asked without prejudice to the Concrete treatment subjects, and also to round out the treatment in terms of what is considered important outcomes of the learning. This may be seen more clearly after the embodiments chosen have been described as parts of the other treatments, so phase II embodiments will be dealt with again in more detail at those points. These phase II embodiments are dealt with in an entirely different way than the embodiments under phase I of the treatments. Under the assumption that the structure and its symbolization have been learned during phase I, none of the principles of the learning of the structure on which phase I was based need be adhered to. Hence these two embodiments are merely presented in an expository manner whereby outcomes of combinations of pairs of moves are presented in lattice table form, the manner of arriving at these is stated if possible, and exercises are provided for practice in performing the combinations. The actual Primary and Secondary Symbolic embodiments used in the second phase of the Concrete treatment will be identified and discussed at the time when the other treatments are discussed later in this chapter.

The Primary Symbolic Treatment

Besides utilizing the same general principles as for the Concrete treatment this treatment makes use of the Schematic Principle and rejects the Physical and Dynamic Symbolization Principles. In accord with the Schematic Principle the learning of the Klein and cyclic group structures is built on what should be considered familiar schemas by the subjects. If these are secondary concepts or primary ones that have been symbolized, then symbols must be used. One embodiment used in phase I of the Primary Symbolic treatment is modular 4 arithmetic. It is this embodiment that is also used in phase II of the Secondary Symbolic and Concrete treatments. In accord with the free play stage of the Dynamic Principle subjects were asked to use the four basic arithmetic operations on pairs of numbers 0 to 4 in as many different combinations as they wished. For the structured game stage the children were taught addiv, which is finding the remainder on adding a pair of numbers and dividing by a given number. For example, when the divisor is 4 and the addends are 2 and 3 we have: 2 x 3 = 5, $5 \div 4 = 1$ with remainder 1, so $2 \oplus 3 = 1$, where $\oplus$ is the symbol used for addiv. Another way of expressing this, which indicates that the divisor is 4, is $2 \oplus 3 = 1 \bmod 4$.

The first card used for this structured game is as follows:

Addiv Game

1. What can you addiv to 1 to make the answer

1? $(1 \oplus \_ = 1)$.

2. If you addiv 0 to 0 what is the answer? $(0 \oplus 0 = \_)$.

3. What can you addiv to 2 to make the answer 2? $(2 \oplus \_ = 2)$.

4. What can you addiv to 3 to make the answer 3? $(3 \oplus \_ = 3)$.

This card focusses on the addiv identity property of this group. Other cards, focussing on other properties are presented in Appendix A. As in the Concrete treatment the subjects were to work in groups of three or four and would be encouraged to communicate among themselves. Data was to be recorded in tabular form as for the Concrete treatment, but in symbolic rather than word form.

A sample practice game card for this treatment is as follows:

Triple Identity

1. Take turns as dealer. Dealer use the equation table and write an addiv equation which shows a three number addiv expression equal to a two number addiv expression.

2. Player must replace the two number expression on the right by an equal three number expression.

3. Verification is by the equation table or carrying out the addiv operations involved.

4. Scoring:

    a) If replacement is identical to three number expression on left - 1 point.
    b) If replacement uses same numbers as expression on left but in different order - 4 points.
    c) If replacement uses at least one zero when it is either not in the number on the left or not in a different position from the one on the left - 6 points.

d) If replacement contains different numbers in all positions from that on the left and at least two numbers of the replacement are not on the left - 8 points.

5. Your group must score at least 40 points before trying another game. You must change games after you have scored 60 points.

The essential aspect of the Constructivity Principle is that abstraction of concepts should be made possible through manipulation of that to which the concept refers. According to Skemp the group structure involves basic properties which are secondary concepts. The modular and arithmetic embodiment of the cyclic group of order 4 presented here is expressed in terms of symbols, and thus it is these that must be manipulated in building up exemplars from which the group structure may be abstracted. Another aspect of the Constructivity Principle is that construction must precede analysis. This is accomplished in the Primary Symbolic treatment in the same way as in the Concrete treatment. All experiences are designed, as mentioned above, to foster abstraction through manipulation. Verbalization or attention to analytical procedures are never requested of the subjects although the opportunity for analysis is provided for those who are ready, able and disposed to adopt it.

The other principles, Contrast and Multiple Embodiment are also provided for in this treatment in the same way as for the Concrete treatment. However another primary symbol game embodiment of the cyclic group is used in place of the story embodiment. The story embodiment was included in the Concrete treatment to take into account the modification of the Physical Principle whereby increasing experience of manipulation of mental imagery replacing manipulation of physical objects is called for. Since the Physical Principle is not to be adhered to in the Primary Symbolic treatment, it was not necessary to include story embodiments.

The unique basis of this treatment, the Schematic Principle was incorporated by means of the choice of embodiments. For instance, the modular 4 arithmetic embodiment is based on the familiar schemes of addition of whole numbers of division with remainders. The semi-concrete nature of the symbolic representation of the concepts to be abstracted is in accord with the Schematic Principle.

For the second phase of the Primary Symbolic treatment the Symmetries of a Rectangle embodiment and one Secondary Symbolic treatment embodiment were presented in an expository manner and followed up with practice exercises.

The Secondary Symbolic Treatment

This treatment was also built on the five basic Dienes' Principles, Dynamic, Constructivity, Contrast, Multiple Embodiment and Mathematical Variability. The Physical and Dynamic Symbolization Principles were rejected and replaced by the Secondary Symbolic Principle. According to this principle it is possible for children to form the basic concepts of the group structure by engaging in appropriate activities involving meaningless symbols so long as these activities are designed to emphasize the basic group properties such as associativity, inverse elements, etc.

For the free play stage the children were separated into small groups of three or four. They were each given a sheet of paper on which all the symbols of the four embodiments used were shown. Each group was also given a set of cards showing pairs of symbols on one side and a single symbol on the other. Each player in the group took turns at being in charge of the cards. It was his responsibility to display the side of the card with two symbols on it on request from any students in his group, and then later, on request from the same student, to display the single symbol side. The other students in the group were asked to practice copying the symbols on another piece of paper. These had to be copied in strings of three. There was completely free choice in the first two symbols chosen for each string of three, but the third one had to be the one on the back of the card containing the other two symbols.

The first structured game restricted the number of symbols to four for all but the few experiences designed to help students in abstracting the closure property. A card used for highlighting this property in the structured game stage is as follows:

Guess Back

1. Use the cards with the printed S1A at the top left corner.
2. Spread them out on the table in front of you so that the side with the two symbols with an upward arrow like this ↑ between them is showing.

3. Try to guess which of the following symbols is on the back of any card you pick: W, 8, ⊙, =, T, ✓.

4. Ask someone in your group who is working on a S1B card to check the backs of your cards to let you know if you are right in your guess. (Check the cards of other people in your group if they ask you).

The symbols T and ✓ appeared on the front of several cards but not on the back of any. At this stage the subjects were required to construct the same two charts as made for the other treatments for later analysis.

The third necessary stage according to the Dynamic Principle, the practice game is illustrated for this treatment by the following game:

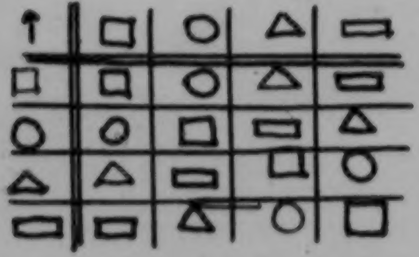
Wipe Out

1. Without using the lattice table, each person take a turn at laying down a card in a row.

2. Player now checks the lattice card. If your card and the card next to it give a □ in the lattice table pick up all cards laid down and begin the game all over again.

3. When all cards have been layed down the game ends unless the last pair make the □, in which case the game must be begun again.

The lattice table for this embodiment is shown in figure 6.



↑	□	○	△	▭
□	□	○	△	▭
○	○	□	▭	△
△	△	▭	□	○
▭	▭	△	○	□

FIGURE 6

Lattice Table for a Secondary Symbolic Embodiment

This is the Klein 4 group in which every element is its own inverse. As can be seen from this table the game highlights the inverse property. This embodiment is presented in phase II of the other two treatments.

The Constructivity Principle is adhered to by arranging the activities so that subjects manipulate symbols in such a way that exemplars of the group structure properties are created and thus available for abstraction. Analysis is made possible by continued use of the charts which may reveal patterns to those subjects ready, willing and able to undertake analysis.

As with the other two treatments all symbols are used many times, four embodiments are used, and activities highlighting non-exemplars of properties are included in accord with the tenets of the Mathematical Variability, Multiple Embodiment and Contrast Principles respectively.

For the phase II portion of this treatment the Symmetries of a Rectangle and Modular 4 Arithmetic embodiments are presented in an expository way and followed by practice exercises.

The Criterion Measures

The purpose of the criterion measures, known as the Group Concepts tests, was to compare the mean scores on the learning of the group structure properties under the three treatments. The four tests were based on Dienes' ideas on mathematics and learning. He views applied and pure mathematics as equally important. Concerning the pure side, he has suggested that development of the ability to symbolize and interpret are major ultimate aims. But the real test of learning structural relations for both he and Skemp is transfer. Thus the four tests used are Symbolization, Interpretation, Transfer and Application. The test-retest reliability established for the combined four tests in the pilot studies is 0.81. The standard deviation for the combined four tests is 6.37 and the readability as calculated by the Flesch formula is grade seven level.

At this point a particular example of an item from each test should serve to clarify the meaning of the terms. The following is an item designed to test symbolization of the commutative property.

> Using any of the letters a, b, c, d or e to represent movements of the happy face card:
>
> 1. Express the idea that when the order of doing any pair of movements is reversed the happy face ends up in the same place.

The Symmetries of a Rectangle group was used in all treatments, and the commutative property was presented in phase I of all treatments under several embodiments. Hence this item tests whether a subject has abstracted the commutative property sufficiently well so that given a familiar non-symbolic context it can be symbolized by him.

An example of an interpretation item is as follows:

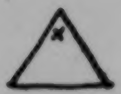
> Let the symbols A, B, C, and D represent the numbers of Mod 4 Arithmetic. Which of the following are always true for Mod 4?
>
> a) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

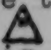
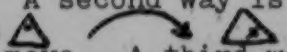
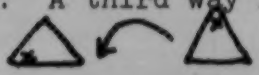
b) $A \oplus C = C \oplus B$
c) $A \oplus (C \oplus B) = (B \oplus C) \oplus A$
d) $(A \oplus A) \oplus D = C \oplus (A \oplus D)$

Give at least one example of each of the above that is true.

This item tests knowledge of the commutative and associative properties and interpretation in terms of modular 4 arithmetic.

The transfer test contains items concerning embodiments similar to those studied but nevertheless novel as far as the treatments are concerned. Most of the items chosen are from examples by Dienes (1967) of groups other than the Klein and cyclic group of order four.



If we want the base of the triangle shown to be any side that is horizontal there are three different ways to accomplish this. The first is to leave the triangle as it is with the x vertex at the top:  . This we call the NO CHANGE move. A second way is to flip it to the right as shown:  . This will be called the right move. A third way is to flip it to the left as shown:  . This is known as the LEFT move.

Show the similarity between the modular 4 arithmetic and these moves by (1) stating the moves which express the same ideas as $1 \oplus 3 = 0$, and $2 \oplus 2 = 0$.

The idea here is to show recognition of the inverse property of the cyclic 4 group in the context of the motions of the equilateral triangle which form a group of order 3.

The requirements of the application category are that the item involve the known cyclic 4 or Klein structure, but that the embodiment be unfamiliar. Such an item is as follows:

. . . Suppose we have two light switches. Starting with both lights off we can switch the left, or we can switch

> the right, or we can switch both together, or we can switch neither. For example, starting with neither on, then switching the left turns the left light on. Then throwing the left switch again turns the left light off, so it amounts to the same thing as throwing neither switch. If this is to be compared to the motions of the rectangular happy face card,
>
> 1. is there an equivalent to the idea that any motion followed by a 360° rotation is the same as just that motion alone? If so, give at least two examples.

The property of the identity element of a group is being tested here. The notion of an identity element for the group structure embodiments encountered in the treatments serves as the basis of the question. Recognition of this property in the context of the unfamiliar Klein group involving light switches constitutes the application.

CHAPTER V

THE RESULTS OF THE INVESTIGATION

This chapter consists of the presentation of the results of the data analysis. In the first section a comparison of the means of the three treatments on the criterion tests by one way multi-and uni-variate analyses of variance and multiple comparisons is reported. This analysis is discussed with reference to null hypothesis 1. The second section consists of the presentation of three separate two-way multivariate analyses of variance. Each of these three analyses deals with a different dichotomized independent variable, reflective intelligence, arithmetic achievement and sex respectively. The main effects of each of these is tested by multivariate analysis of variance, and significant differences between mean vectors are further analyzed by multiple comparisons. In addition, for each variable, interaction among treatments and the categories of the variable is tested by multivariate analysis of variance. The results of each analysis are discussed with respect to the appropriate null hypothesis. For all analyses the criterion is the Group Concepts test battery consisting of tests of Symbolization, Interpretation, Transfer and Application.

Comparison of Treatments

In order to decide between multi-and univariate analyses a multiple correlation study was carried out. The results of this study, summarized in Table I, suggest that the four criterion tests were not independent so the multivariate analysis would be more appropriate. The procedures used for carrying out the multivariate analysis of variance are those recommended by Morrison (1967). The null hypothesis tested is as follows:

> H_1 There is no difference among treatment mean vectors of the Group Concepts tests.

TABLE I

Correlations Between All Possible Pairs of Criterion Tests

	Symbolization	Interpretation	Transfer	Application
Symbolization	1.000	0.410	0.585	0.415
Interpretation	0.410	1.000	0.491	0.338
Transfer	0.585	0.491	1.000	0.404
Application	0.415	0.338	0.404	1.000

Table II is a list of mean scores on the Group Concepts tests for each treatment. Table III is a summary of Rao's F-test for multivariate analysis of variance on the treatment mean vectors of the Group Concepts test.

Since the probability is 0.0835 there is no significant difference among treatment group mean vectors at the 0.05 level of significance. Hence, the null hypothesis is not rejected.

TABLE II

Group Concepts Tests Cell Means for the Three Treatments

Test	Treatments		
	Concrete	Primary Symbolic	Secondary Symbolic
Symbolization	5.04	5.05	5.54
Interpretation	4.64	4.30	5.14
Transfer	4.88	4.13	4.84
Application	4.18	3.48	4.11

TABLE III

Summary of Multivariate Analysis of Variance on Treatment Mean Veetors, Rao's F-test

Source	df1	df2	F	P	Wilks Lambda
Treatment	8	318	0.18	0.0835	0.9168

However, in view of closeness to significance at the 0.05 level, multiple comparisons were carried out. These revealed no significant differences between pairs of treatments close to the 0.05 level, the closest being between the Primary Symbolic and Secondary Symbolic treatments at the 0.58 level on Interpretation. Since multivariate analysis of variance is a relatively conservative procedure which can mask true differences it was decided to use the less conservative procedure of univariate analysis of variance on the Interpretation test. This was done only with a view to discussing implications for further research. The correlation between pairs of criterion tests indicates that valid conclusion may be drawn only from a multivariate analysis. The procedures used for the analysis of variance are those recommended by Winer (1962). The analysis is summarized in Table IV below.

TABLE IV

Summary of Analyses of Variance for the Three Treatments on the Means of the Group Concepts Tests

Test	Source	SS	df	MS	F
Symbolization	Treatment	9	2	4.50	1.07
	Error	677.3	161	4.21	
Interpretation	Treatment	19.9	2	9.85	3.34*
	Error	474.2	161	2.95	
Transfer	Treatment	20.1	2	10.05	2.07
	Error	780.2	161	4.85	
Application	Treatment	16.4	2	8.20	2.34
	Error	563.9	161	3.50	

Decision rule: Reject hypothesis if Fobs $\geq$ 3.06 $[F_{crit\ 0.95\ (2,161)}]$

As can be seen from Table IV the observed F-ratio for the Interpretation test exceeds the critical value at the 0.05 level of significance. Since three treatments were involved, tests of significant differences among ordered treatment means were carried out. In Table V the treatment means on the Interpretation test are ordered from least to greatest as the table is read from left to right or top to bottom. The two entries in each cell represent the absolute difference between the means of the treatments compared. Enclosed in parentheses is the critical five percent level of significant differences. If the observed absolute difference exceeds the critical value, the difference is statistically significant at the 0.05 level. This is the Newman-Keuls procedure as described by Winer (1962).

TABLE V

Tests of Differences Between Ordered Mean Scores for the Three Treatments on the Interpretation Test

Treatment Means	Primary Symbolic 4.30	Concrete 4.64	Secondary Symbolic 5.14
4.30	-	0.34 (0.64)	0.84* (0.76)
4.64		-	0.50 (0.64)
5.14			-

Though the results of the Newman-Keuls test indicate that there is not a significant difference at the 0.05 level between Secondary Symbolic and Concrete treatment means, the former is higher. This test does however, indicate that the mean score of the Secondary Symbolic treatment group was significantly higher than the mean of the Primary Symbolic treatment group on the Interpretation test. Though significant differences are indicated here by the analysis of variance, the more conservative test by multivariate analyses indicated that the original null hypothesis of no treatment effects should not be rejected for a 0.05 level of significance test. This analysis shows that it is only with a less conservative test which does not take into account the intercorrelations between the criterion tests, that the null hypothesis may be rejected.

Comparison of Categorization Variables

Multivariate analyses of variance and multiple comparisons were used to determine interaction between treatments and categorization variables of reflective intelligence, arithmetic achievement and sex, as well as main effects of these three variables. The mean scores for each category by treatment on the Group Concepts tests are shown in Table VI. Table VII is a summary of Rao's multiple analysis of variance test, and Wilk's Lambda, on the categorization variable and treatment by categorization mean vectors.

Interaction Effects

The null hypotheses tested in order to determine interaction are as follows:

H_5 There is no interaction among treatment and reflective intelligence category mean vectors of Group Concepts tests.

H_6 There is no interaction among treatment and arithmetic achievement category mean vectors of Group Concepts tests.

H_7 There is no interaction among treatment and sex category mean vectors of Group Concepts tests.

The results of Rao's F-test as shown in Table VII indicate no interaction whatsoever. Therefore, none of the three interaction null hypotheses is rejected.

The results may have been different if some other procedure of establishing high and low categories had been used, such as selecting the top and bottom thirds, rather than selecting those above and below the median.

TABLE VI

Group Concepts Tests Cell Means for Each Category by Treatment

Test	Category		Treatments		
			Concrete	Primary Symbolic	Secondary Symbolic
Symbolization	Reflective	High	6.07	5.75	6.11
	Intelligence	Low	4.00	4.36	4.96
	Arithmetic	High	6.00	5.82	6.68
	Achievement	Low	4.07	4.29	4.46
	Sex	Boy	4.32	4.50	5.46
		Girl	5.75	5.61	5.61
Interpretation	Reflective	High	5.18	4.96	5.28
	Intelligence	Low	4.11	3.64	5.00
	Arithmetic	High	5.32	5.21	5.50
	Achievement	Low	3.96	3.39	4.79
	Sex	Boy	4.89	3.96	5.25
		Girl	4.39	4.64	5.04
Transfer	Reflective	High	5.61	4.68	5.07
	Intelligence	Low	4.14	3.57	4.61
	Arithmetic	High	5.39	4.89	5.68
	Achievement	Low	4.36	3.36	4.00
	Sex	Boy	4.68	4.00	4.68
		Girl	5.04	4.25	5.00
Application	Reflective	High	5.00	3.89	4.12
	Intelligence	Low	3.36	3.07	4.12
	Arithmetic	High	4.43	3.68	4.61
	Achievement	Low	3.93	3.29	3.61
	Sex	Boy	4.25	3.36	4.25
		Girl	4.11	3.61	3.96

Categorization Variable Effects

As shown in Table VII there are significant differences between high and low categories for reflective intelligence, arithmetic achievement. The null hypotheses tested were as

TABLE VII

Summary of Rao's F-Test for Multiple Analysis of Variance on Category Main Effects and Treatment by Category Interaction

Source	df1	df2	F	P	Wilk's Lambda
Reflective Intelligence	4	159	6.48*	0.00007	0.859745
Arithmetic Achievement	4	159	11.37*	0.0	0.777553
Sex	4	159	2.52*	0.04336	0.9404
Treatments x Reflective Intelligence	8	318	0.95	0.47	0.953647
Treatments x Arithmetic Achievement	8	318	0.90	0.52	0.956133
Treatments x Sex	8	318	1.10	0.37	0.947281

follows:

H_2 There is no difference between reflective intelligence mean vectors of the Group Concepts tests.

H_3 There is no difference between arithmetic achievement mean vectors of the Group Concepts tests.

H_4 There is no difference between sex mean vectors of the Group Concepts tests.

All three null hypotheses were rejected at the 0.05 level of significance and multiple comparisons were then made for these. The comparisons are shown in Table VIII.

The multiple comparisons shown in Table VIII reveal significant differences at the 0.05 level for the Symbolization and Interpretation means for both reflective intelligence and arithmetic achievement, and also for Transfer in the case of arithmetic achievement. Inspection of the cell means shown

TABLE VIII

Multiple Comparisons Using Rao's F-Test for the Three Categorization Variables on Groups Concepts Tests

Categorization Variable	Test	df1	df2	F	P
Reflective Intelligence	Symbolization	4	159	5.46*	0.00038
	Interpretation	4	159	2.60*	0.04
	Transfer	4	159	2.09	0.08
	Application	4	159	2.10	0.08
Arithmetic Achievement	Symbolization	4	159	8.61*	0.0
	Interpretation	4	159	5.93*	0.00018
	Transfer	4	159	4.29*	0.00251
	Application	4	159	1.18	0.32
Sex	Symbolization	4	159	1.70	0.15152
	Interpretation	4	159	0.004	1.0000
	Transfer	4	159	0.17	0.95287
	Application	4	159	0.01	0.99979

in Table VI indicates that the differences favor the high categories in all cases.

Summary

The only differences observed at the 0.05 level of significance were in the three categorization variables. Of these only two, reflective intelligence and arithmetic achievement showed significant differences on individual tests. All differences favored the high categories on Symbolization and Interpretation for both reflective intelligence and arithmetic achievement, and also for Transfer in the case of arithmetic achievement only. Though no significant differences in treatment effect were observed at the 0.05 level, there was a difference at the 0.08 level.

Further analysis by less conservative procedures, not normally appropriate when the criterion variables are not independent as in this case, revealed this difference to be in the Interpretation test for the Secondary Symbolic treatment over the Primary Symbolic.

CHAPTER VI

SUMMARY, CONCLUSIONS AND IMPLICATIONS

In the first section of this chapter the experiment is briefly summarized. This is followed by the presentation and discussion of conclusions drawn from the two sources. The first source is the development of this study and its experimental results. The second source is works of Dienes published subsequent to this study. In the final section implications for practice, theory and further research are presented and discussed.

Summary

Consideration of Dienes' ideas on mathematics education led to the development of three approaches to the learning of mathematical structures. All three approaches were based on a set of several principles proposed by Dienes, but differed from each other in principles concerned with the role of symbols. From these approaches initial phases for each of three treatments were developed. Phase I of the Concrete treatment followed all Dienes' Principles as closely as possible, and made use of the Physical and Dynamic Symbolization Principles as unique aspects of the treatment. The unique aspect of the Primary Symbolic treatment was the Schematic Principle, and the Secondary Symbolic Principle constituted the unique part of the Secondary Symbolic treatment. Phase II of each treatment consisted of a brief expository exposure to two embodiments of group structure, one from each of the other two treatments.

In the experimental part of this study the effects of these treatments on symbolization, interpretation, application and transfer for grade seven students was studied. In addition the variables of reflective intelligence, arithmetic achievement and sex differences were examined. Interaction among each of these variables and the treatments on the four criterion tests was also studied. The mathematical content of the treatments were the Klein and cyclic four groups.

The results showed no treatment effects at the 0.05 level of significance under multivariate analysis, but a difference at this level favoring the Secondary Symbolic over the Primary Symbolic treatment on the Interpretation test means using the less conservative analysis of variance techniques. There were significant reflective intelligence, arithmetic achievement and sex differences at the 0.05 level, but only the former two showed differences on individual tests. For reflective intelligence the means on Symbolization and Interpretation tests for the high group were significantly higher than for the low group. For arithmetic achievement the same result was observed, but in addition the mean of the high group on the Transfer test was significantly higher than that of the low group.

Conclusions

The more conservative multivariate analysis leads to the conclusion that none of the treatments is superior. However, the fact that the less conservative test shows superiority of the Secondary Symbolic over the Primary Symbolic treatment at least warrants further discussion, as there may be implications for further research.

There are sex differences in the learning of group structures at the grade seven level. However, the difference, which favours girls, does not show up for the individual tests. There are also reflective intelligence and arithmetic achievement differences favoring the high groups in both cases. Though all these differences are related to the theory, and will be discussed as such, the pattern of results for the individual tests is not so clear. In the discussion to follow, an attempt is made to find the most all embracing theoretical explanation of the results that is possible.

Discussion

Concerning the result of no differences in treatment effects one possible explanation is related to the nature of the treatments themselves. Because three embodiments of the group structure were common to all three treatments it might be hypothesized that the treatments were equivalent with respect to learning the group structure. This is extremely unlikely. In the first place, twelve class periods were devoted to the embodiments uniquely characterizing the first phases of each treatment, compared to one period for the adding on of phase II embodiments. In the second place, there was a special attempt to aid student learning of the structure using Dienes' Principles in the first phase, whereas structure was not emphasized in phase II. In this phase memorization was emphasized more than anything else.

Though there were no treatment effects under the particular test conditions selected for this study, it may not be concluded that the treatments were acting in the same way. The three treatments were producing outcomes measured in this study which did not differ significantly. However, the treatments may have been producing different results that were not directly measured in this study. These are of interest, if indeed they occurred, only in so far as they are related to the theory of the approaches, and, in particular, the role of symbols. It is for this reason that the treatment differences in the Interpretation test warrant further examination. If this result can be attributed to some differences in the treatments, and these same differences may be used to explain the reflective intelligence, arithmetic achievement and sex results, then the all embracing explanation sought will have been achieved.

The starting point for the examination of the Interpretation test result is an examination of the interpretation process. As used in this study, interpretation means, given a structure symbolically represented, an embodiment of the structure, which assigns meaning to the symbols may be produced. For instance, given the symbolic representation of the commutative property of the group structure, $a.b = b.a$, an embodiment such as $1 + 2 = 2 + 1$, or horizontal flip of rectangular card followed by vertical flip is equivalent to vertical flip followed by horizontal flip is produced as an interpretation. In order for this interpretation to be made the structural relation of commutivity must have been abstracted at least to a limited extent. In the first place, the interpreter has to make sense of the unfamiliar symbolic representation. The pattern, or structural relation, must be detected in order to search among known representations. In the second place the structural relation must be known to apply to embodiment meaningful to the interpreter. This indicates at least a limited amount of abstraction, as the structural relation must refer to both an initially meaningless embodiment and a meaningful known embodiment. If the learner is able to abstract to this limited extent, then he is capable of interpretation.

The following two conjectures are consistent with the experimental results and each other.

> Conjecture 1: The learning context of meaningless symbols in the Secondary Symbolic treatment facilitates the reaching of a minimal level of abstraction necessary for interpretation more than either of the other treatments do.

The Concrete treatment is included in this comparison because for interpretation the mean of the Secondary Symbolic treatment was higher, though not significantly at the .05 level.

Conjecture 2: The two processes symbolization, and transfer require a higher level of abstraction than interpretation does.

This latter conjecture is consistent with the findings of reflective intelligence and arithmetic achievement differences. It is putting these other three processes on a higher cognitive plane than interpretation. In order to be completely consistent it might be speculated that transfer is in turn on a higher plane than symbolization. This is quite reasonable because symbolization requires only familiar embodiment knowledge whereas some contextual novelty is required for transfer. Thus it is reasonable to suppose that transfer requires a higher level of abstraction, or some other process such as generalization, or both. Though the sex finding cannot be incorporated into the explanation for the other results, it is in line with other findings of Dienes in his study of Concept Formation and Personality. He found that girls tended to approach learning tasks from the point of view of the whole structure, whereas boys tended to be more analytical. Since the learning tasks of all treatments were constructively loaded the girls should have been at an advantage then.

Dienes' Most Recent Findings

Subsequent to carrying out this experiment several relevent works of Dienes have been published. In one of these Dienes (1970) describes an experiment to determine the effects of structural relations on transfer. This study is of particular significance because it involved the learning of group structures in the context of meaningless symbols. In addition, by using both children and adults it was possible to study developmental aspects. The fact that learning in the context of meaningless symbols produced sufficient abstraction of the group structure to permit transfer is of interest. Of even more interest is Dienes' description of the types of generalizations involved in the transfer tasks, and the finding that children could particularize as well as adults, but not generalize so well. In addition children appeared to transfer best from more complex to less complex structures. This, Dienes refers to as the Deep End Principle. These results are not inconsistent with the results and conclusions of this study. Dienes' findings that an initially meaningless symbol context of learning can lead to at least a limited type of abstraction of mathematical structures, and that other conditions such as generalization are necessary for transfer between structures, fit the explanation of what was happening in the Secondary Symbolic treatment.

In another publication of Dienes (1973), he has reformulated some ideas in a different way. He has expanded on the Dynamic Principle and incorporated Dynamic Symbolization and an axiomatizing stage into six stages of learning mathematics. The first three stages are what formerly constituted two of the stages of the Dynamic Principle, namely the play and structured game stages. The fourth stage comes from Dynamic Symbolization and consists of representation of the structure in graphical or some other form. From Dienes' description it appears that it is at this stage that interpretation first becomes possible. In his summary of the stages he states: "The individual becomes able to fill into empty representation those states and operators of a particular game from the structure in question." (p. 53). This statement describes interpretation as used in this study. The fifth stage is that of analysis whereby the properties of the representation are studied. It is at this stage that symbolization is needed. The sixth and final stage is that of axiomatization and proof.

The only aspects of Dienes' theory directly tested in this study were his ideas on concrete and symbol learning contexts. His notion that structural relationships must be introduced first through concrete embodiments and then followed by a gradual weaning from concrete to symbolic modes of representation was not supported by this study. This conclusion is based on the result of no significant differences among the three treatments on the multivariate analysis. Recall that only one of the three treatments was based on Dienes' ideas on concrete and symbol learning contexts.

There is evidence of support, though less formal, for the other major ideas of Dienes' theory which were incorporated into all three treatments. The two instructors of the experiment, one of whom was the investigator, were also the regular mathematics classroom teachers for all subjects. In their opinion, the achievement of subjects in the experiment was at least comparable to that accomplished in the regular mathematics program. Therefore, these other ideas of Dienes, such as the Multiple Embodiment Principle, taken as a whole, were indirectly supported by this study, by virtue of the fact that reasonable achievement was produced from treatments based on these ideas.

Limitations

This study is limited by the fact that some areas that may be related to the problem were not considered. First it should be noted that some studies similar to this one have explored the relationship between attitude and learning. However, since Dienes has not formally incorporated attitude into his theoretical pronouncements, it was excluded from the present study in an attempt to limit the breadth of the investigation. Similarly, retention has not been emphasized by Dienes, and was therefore omitted from this study in order to narrow its scope. It is conceivable that a pattern of results different from that of this study might have been derived from criterion tests administered several weeks after the completion of the treatments.

Two additional limitations should be mentioned. One relates to the readability of the criterion tests. Since these appear to be highly verbal, the subject scores may be related to their reading ability. The other concerns generalizability of results. The reader is cautioned against any attempts to generalize the results of this study beyond the population from which the sample was drawn.

Implications

For Practice

It is not likely that many seventh grade mathematics programs will include learning of formal group structures as part of core content. However, it is quite conceivable that such content would be chosen for enrichment. The implications of the results of this study are that when such content is used for enrichment purposes, those students who have done well in basic arithmetic are the most likely to learn the structures. Less formal structures or parts of formal structures are likely to be included in core content. When this is the case, the results of this study imply that any one of the three approaches described in this study could be adopted, since the outcomes are comparable in terms of the criterion tests used in this study.

For Theory

The results of this study indicate that the unique principles concerning the role of symbols for the three approaches are questionable. It seems likely that a more comprehensive principle or set of principles is needed. These might be derived from a further consideration of the roles of such basic thought processes as abstraction and generalization. From this study it appears that abstraction plays a crucial role in the learning of the properties of a formal structure. From recent works of Dienes (1973) it can be concluded that generalization and other structural relations such as particularization and embeddedness are important in transferring knowledge of structures to the learning of new structures. It may be hypothesized that meaningless symbol learning contexts promote at least a limited type of abstraction, enough for interpretation to be possible. However, either a higher level of abstraction, or other mental processes such as generalization, or both are needed for symbolization and transfer.

For Further Research

The finding that there were no significant differences in treatment effects warrants further investigation. The treatments were substantially different with respect to the role of symbols, yet the only difference in outcomes as measured in this experiment was inconclusive. This was the finding of differences in interpretation. Further research into this difference should be conducted. This should take the form of determining the extent of abstraction necessary for interpretation and the other criterion processes. In addition, the effects of the three learning contexts of this experiment on degrees of abstraction should be the subject of investigation. It might be useful to conduct these experiments for different age levels so that developmental effects might be noted. It might also be useful to select content from other formal mathematical structures, and informal ones too, so that the results might be generalizable over a broader mathematical content area.

As is suggested in the paragraph above, a variety of approaches to the study of the role of symbols in mathematics learning are possible. Learning theory bases, instructional approaches, mathematical content, and learner characteristics different than those used in this study should be selected. However, particular results of this study, the findings of no treatment effects nor interactions, suggest that a more exploratory approach might be most fruitful. One possible form of such an approach consists of studying normal teaching and learning settings. Care is taken not to interfere with the normal situation as far as possible within the framework of the experiment. The investigator acts as an observer, interviewer and analyst. First he formulates a number of questions concerning the role of symbols in mathematics learning that he thinks might be answered through such an experiment. Through observation of lessons, study of tapes, post lesson interviews with students and teachers, and analyses of these, the investigator describes relevant situations which contribute to complete or partial answers to the questions he has raised.

Information provided in this report should be of use to researchers planning future investigations into the role of symbols in mathematics learning.

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APPENDIX A

INSTRUCTIONAL PROCEDURES

Each treatment group met separately for three forty minute sessions each week over four weeks, and progressed through a similar sequence of experiences. The two instructors discussed each session fully beforehand, and both were always present at each session in all treatments.

Phase I

Each treatment group was subdivided into working groups of three to four students for all activities of this phase. In the free play and structured game stages, each treatment group experienced four embodiments of mathematical groups of order four in the following sequence: Klein, cyclic, Klein, cyclic. Under each embodiment in the structured game stage each of the five group properties closure, commutivity associativity, inverse elements and identity element were dealt with but not necessarily in the same order for each treatment. The actual embodiments differed among treatments in order to emphasize the different roles assigned to symbols in the treatments.

At the outset of each embodiment an instructor gave verbal directions for free play activities. The two instructors divided this and all other presentations equally between them. In this particular case each instructor gave the directions for one Klein and one cyclic group to each treatment group.

For the structured game stage students were given cards outlining the games for each embodiment. Both instructors worked with the groups of three or four students in clarifying directions. Time of movement on to a new embodiment varied among treatments, but within each treatment the changeover of the two required charts and evidence of understanding the games (as judged by an instructor) for all students of the treatment.

For the practice games the same four embodiments were repeated in a conventional game format in the same order as previously encountered. Each game had a built-in mechanism for progressing to the next game or embodiment, based on points scored. The instructors informally checked each student's mastery of these games, and insisted on continuation of the practice stage until all students had mastered the games. This resulted in four students in the Concrete treatment being re-organized into a separate working group for the eleventh session and playing the Wipe Out game until the instructor felt that each of these students understood it sufficiently.

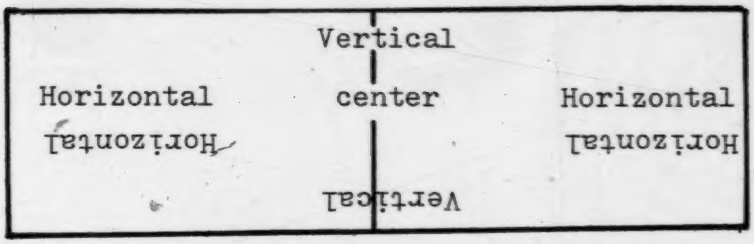
Phase II

The investigator made all the presentations of this phase during the twelfth and last forty minute session. During the initial twenty minute lecture of the session, the students were presented information on one embodiment from each of the other two treatments. First the embodiment was briefly described and then the outcomes of each pair of moves was recorded on the blackboard in the form of the two charts. In the case of the Concrete embodiment, for example, the instructor wrote each combination on the blackboard, demonstrated it with the happy face card, and required each student to perform the move also. For the mod 4 embodiment the instructor gave some sample addiv calculations and required each student to furnish additional ones. Each of the five group properties was demonstrated and explained by the instructor for each embodiment presented. For the final twenty minutes of the session, students were required to reconstruct the charts as often as possible.

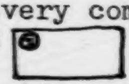
SAMPLE OF TREATMENT CARDS

Concrete Treatment

Card C1 **Happy Face Card**



Card C2 **Happy Face Card Directions**

1. The starting position for every combination of motions is always like this  with the happy face at the top left. From this starting position the following motions may be combined:

2. Horizontal. Hold each end of the horizontal line and flip the card over onto its back.

3. Vertical. Hold each end of the vertical line and flip the card over onto its back.

4. Half Turn. Place the card on a flat surface and rotate it around the center point until it is upside down from its previous position.

5. Whole Turn. Place the card on a flat surface and rotate it around the center a complete turn back to its previous position.

Card C3-2 **Diagonal Game**

1. Draw a line from the bottom left hand corner to the top right hand corner of your happy face card.

2. Place the card on the desk in front of you in the normal starting position (happy face at top left).

3. Begin to turn the card about the center as you would for a half or whole turn but stop when the line you drew is parallel to the edge of the desk in front of you. This is called the quarter turn move.

4. Can you make any pairs of moves that would leave the happy face in the same place as making just the quarter turn move?

5. What single move is the same as two quarter turn moves?

6. Is there any single move that puts the happy face in the same place as three quarter turn moves?

Card C3-3

Biflip Game

1. A horizontal flip then vertical flip brings the happy face to the same place as a half turn. What other pair of motions bring the happy face to the same place as a half turn?

2. Which of the above pairs of motions does not include a whole turn?

3. A horizontal flip then a half turn is the same as a vertical flip. Which other pairs of motions combined is equivalent to the vertical flip?

4. Which of the above pairs does not include a whole turn?

5. A horizontal flip then vertical flip then a half turn is the equivalent to a half turn and a half turn which is equivalent to a whole turn. Is there any other way of expressing the three motions horizontal flip vertical flip and half turn as two motions which are equivalent to a whole turn? Which of these have the horizontal flip as the first of the pair of motions?

Card C5-3

Same Tire

1. Do a diagonal tire rotation on your model car. What other rotation can you make now that leaves the tires where they are?

2. Do a clockwise tire rotation. What other rotation done now leaves the tires where they are?

3. What rotation following a counter clockwise one leaves the tires in the same place as doing the counter clockwise rotation only?

4. Do a no rotation. What happens to the tires if you now do another no rotation?

Card C7-1

Void

1. If you change from squares to circles (circles to squares), which of the other three motions can you now do so that nothing will change?

2. If you change from red to blue (or blue to red), which of the other three motions can you now do so that nothing will change?

3. Write down all the pairs of motions that will result in no change to the colored shapes you have.

C9-2

Walk Around

1. Four people are standing in a room, one at each corner. First they move clockwise to the next corner and then move diagonally to the corner opposite. Which other pair of the four moves would have brought the people to the same corners?

2. Which of the above pairs do not include the no move?

3. A counter clockwise move then a diagonal move is equivalent to a clockwise move. Which other pairs of moves combined are the same as the clockwise move?

4. Instead of doing three moves in a row the same can always be achieved by two moves, and also by one move. For instance, clockwise then counter clockwise, then clockwise again is the same as no move then clockwise, which is the same as just clockwise. What other pairs of moves are the same as these three? Which of these has clockwise as the first move? What single move is the same as counter clockwise then clockwise?

C10-3

Rock and Roll

1. Use your happy face card to play this game in which only the following moves are allowed:

   a) Horizontal Rock - flip about the horizontal line
   b) Vertical Rock - flip about the vertical line
   c) Whole Roll - a whole turn about the center
   d) Half Roll - a half turn about the center

2. Take turns at being the dealer for each different game you play. The dealer starts each play by calling two moves.

3. Each player takes turns at answering. On his turn the player calls two moves he thinks will put the happy face in the same place as the dealer's two moves.

4. Dealer checks the charts to see if player is correct. A game is completed when each player has had a turn.

5. Score:

   a) One point for correct answer if both player moves are different from either dealer move.
   b) Two points if one move is the same.
   c) Three points if both moves are the same but not in the same order.

6. You can pick any new game card and start playing any time after your group has scored a total of 20 or more points at this game.

C11-2

Wipe Out

1. Use your model car and tire rotation moves again for this game.

2. Any one player starts by making any one of the four rotations. A second player tries to make any one of the four moves that will give a wipe out or a pass through. A wipe out is when the tires are moved back to where they were before the first player made his move. A pass through is when the second player's move results in no change from the positions of the tires after the first player's move. After the two moves the second player becomes the first player and a new player becomes the second player.

3. Score:

   Second player scores (1) one point for the team if he successfully does a pass through and names his move correctly, (2) two points if the second player does a wipe out and names his move correctly.

4. Move to new game when total group score is 12 or more.

Primary Treatment

P2-1

Fours

1. On the clock face of card P1 you can only move three spaces from the starting point back to the starting point again. Once you pass the starting point you start counting all over again. Is it possible to move twice for a total of four spaces?

2. If you move three spaces from the starting point, and then one space, is this called a four space move or a one space move?

P2-2

Lost Time

1. Remember all moves are in the clockwise direction except when you follow a move by the same move. In this case the second move is in the counter clockwise direction.

2. Solve the following problems for the moves and clock on card P1:

a) 2 + _ = 2   d) 2 + 0 = _
b) 3 + $\overline{3}$ = _   e) 2 + 2 = $\overline{\ }$
c) 1 + 0 = _   f) 1 + _ = $\overline{0}$

P2-3

Together

1. Parentheses enclosing an expression mean compute that expression before carrying out any computation outside parentheses. Compute the following expressions which refer to making moves on the P3 card clock:

a) (3 + 2) + 1 = 3 + (_ + 1)
b) 0 + (1 + 2) = _
c) (0 + 1) + 2 = _
d) 0 + 3 = _
e) (_ + _) + 2 = 3

2. The expression (2 + 1) + 2 can be written using only two numbers, 3 and 2. Write this expression. What number of spaces on the clock added to 2 is equivalent to 2 + (1 + 2)?

P3-1

Addiv Game

1. What can you addiv to 1 to make the answer 1?
(1 ⊕ _ = 1)

2. If you addiv 0 to 0 what is the answer?
(0 ⊕ 0 = _)

3. What can you addiv to 2 to make the answer 2?
(2 ⊕ _ = 2)

4. What can you addiv to 3 to make the answer 3?
(3 ⊕ _ = 3)

P3-3

Null

Do the following exercises for Mod 4 arithmetic.

1. 2 ⊕ _ = 2   5. _ ⊕ 0 = 3
2. 3 ⊕ _ = 3   6. I ⊕ _ = 1
3. _ ⊕ 0 = 1   7. 0 ⊕ _ = 0
4. _ ⊕ 0 = 2   8. 0 ⊕ _ = 2

P4-1

Muldiv Game

Do the following for the Muldiv 5 using the direction given on card P4.

1. a) $2 \otimes 1 = \_$
   b) $1 \otimes 1 = \_$
   c) $\_ \otimes 3 = \overline{3}$
   d) $\overline{4} \otimes \_ = 4$

2. a) $1 \otimes \_ = 1$
   b) $2 \otimes \_ = 1$
   c) $\_ \otimes \overline{2} = 1$
   d) $\overline{4} \otimes 4 = \_$

P7-3

Triple Identity

1. Take turns as dealer. Dealer use the equation table and write an addiv equation which shows a three number addiv expression equal to a two number addiv expression.

2. Player must replace the two number expression on the right by an equal three number expression.

3. Verification is by the equation table or carrying out the addiv operations involved.

4. Scoring:

   a) If replacement is identical to three number expression on left - 1 point
   b) If replacement uses same numbers as expression on left but in different order - 4 points
   c) If replacement uses at least one zero when it is either not in the number on the left or not in a different position from the one on the left - 6 points
   d) If replacement contains different numbers in all positions from that on the left and at least two numbers of the replacement are not on the left - 8 points

5. Your group must score at least 40 points before trying another game. You must change games after you have scored 60 points.

Secondary Treatment

S1-1

Guess Back

1. Use the cards with the printed S1A at the top left hand corner.

2. Spread them out on the table in front of you so that the side with the two symbols with an upward arrow ↑ between them is showing.

3. Try to guess which of the following symbols is on the back of any card you pick: W, 8, ⊙, =, ✓ T, .

4. Ask someone in your group who is working on a S1B card to check the backs of your cards to let you know if you are right in your guess. (Check the cards of other people in your group if they ask you).

S2-2

Predict

1. Pick out any one of the S2A cards. Write down what is on the front of the card such as □↑○ . Now look at the back of the card. If you picked □↑○ , then ○ would be on the back. Try to find another card with ○ on the back and record what is on the front.

2. Repeat what you did in 1 above several more times with different cards.

3. Now pick out a card you haven't picked before. Look at the symbol on the back. Try to guess which card has the same symbol on the back.

4. Repeat 3 above at least four times using different cards.

S8-2

Wipe Out

1. Without using the lattice table, each person take a turn at laying down a card in a row.

2. Player now check the lattice card. If your card and the card next to it give □ in the lattice table, pick up all cards laid down and begin the game all over again.

3. When all cards have been laid down the game ends unless the last pair make the □ , in which case the game must be begun again.

APPENDIX B

TEST ADMINISTRATION AND SCORING

Directions for writing the tests were given verbally at the outset, with important points written on the blackboard. These included:

1. Work on your own.
2. Answer all questions on the test paper. If you are not sure of the answer, just guess as there is no penalty for guessing (the test copy shown here has been compressed for the sake of brevity, but the student copies provided adequate space for writing answers).
3. Read over all the questions and ask the instructor about anything that is not clear to you before starting the test.
4. Do the questions that seem to be easiest first.

No student was allowed to begin the test until all had indicated they had read all questions and had all queries clarified. Once the test was begun (after initial reading) the time allowed to completion was forty minutes.

The test contains 40 individual items in all, ten for each of the sub-tests, Symbolization, Interpretation, Transfer and Application. Each of these items was scored 1 for correct and 0 for incorrect. There were two items on each property, such as closure, in each sub-test. A summary of the item classification is shown below:

	Group Property				
Sub-Test	Commutative	Associative	Identity	Inverse	Closure
Symbolization	I (1), VIII (3f)	I(4), VIII (1a)	I(2a), VIII (2e)	I(2b), VIII(4g)	I(3), VIII(5d)
Interpretation	III(a), V (2c)	III(d), V (2a)	III(6), V (3a)	III(e), V(3b)	III(c), V(1a)
Transfer	II(4), IV (2)	II(3), IV (5)	II(5), IV (1)	II(1), IV(3)	II(2), IV(4)
Application	VI(1a), VII (2)	VI(2), VII (3)	VII(1c), VII(1)	VI(1b), VII(5)	VI(1d), VII(4)

GROUP CONCEPTS TEST

I Using any of the letters a, b, c, d, or e to represent movements of the happy face card:

1. Express the idea that when the order of doing any pair of movements is reversed the happy face ends up in the same place.

2. Let d mean a whole turn and any three of a, b, c, or e stand for the other motions of the happy face card. Use these letters to express the ideas:

    a) Horizontal then whole turn is equivalent to horizontal, vertical then whole turn is equivalent to vertical, and so on.

    b) Horizontal then horizontal is equivalent to whole turn, vertical then vertical is equivalent to whole turn, and so on.

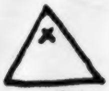
3. Let d mean whole turn and * mean followed by. The other happy face movements are represented by a, c, and e, and the following are true:

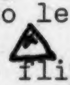
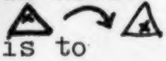
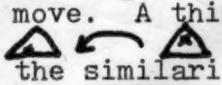
    a*c = e
    a*e = c
    e*c = a

    Therefore, a) b means a half turn
    b) b*d = d
    c) a*a = b
    d) b means one of either vertical or horizontal
    e) b has no meaning for happy face card motions.

4. When two movements are enclosed in parentheses those two must be done first. Using the letters a, c, and e, express the idea horizontal then (vertical then half turn) is the same as (horizontal then vertical) then half turn.

II If we want the base of the triangle shown to be any side that is horizontal there are three different ways to accomplish this.



The first way is to leave the triangle as it is with the x vertex at the top:  . This we call the NO CHANGE move. A second way is to flip it to the right as shown:  This will be called the RIGHT move. A third way is to flip it to the left as shown:  . This is known as the LEFT move. Show the similarity between modular 4 arithmetic and these moves by:

1. Stating moves which express the same idea as $1 \oplus 3 = 0$, $2 \oplus 2 = 0$.

2. In the happy face card flips there were four moves. Given the three moves in the plane LEFT, RIGHT, and NO CHANGE as described above, is there a fourth move that will make one of the sides horizontal without turning the triangle over? If there is such a move it must obey all the same rules of combining moves as the other three moves. If there is such a move, describe it.

3. Movements enclosed in parentheses must be done first. For the triangle flips (LEFT, RIGHT) then LEFT is the same as LEFT then (RIGHT, LEFT). Give an example which shows the same rule for modular 4 arithmetic, or the happy face card motions, or the games you played with the symbols □, ○, △, ▭.

4. A rule of the triangle moves is included in the following true statement: LEFT then (LEFT, RIGHT) is the same as LEFT then (RIGHT, LEFT). Which of the following is an example of the same rule?

   a) $2 \oplus (3 \oplus 1) = 2 \oplus 0$
   b) □↑(△↑○) → (□↑△)↑○
   c) Vertical then horizontal is the same as horizontal then vertical.

5. Another rule of the triangle moves is shown in the example: NO CHANGE then LEFT is the same as LEFT. Which of the following is an example of the same rule?

   a) $0 = 2 \oplus 2$
   b) □↑○ → ○
   c) Whole turn is the same as horizontal then horizontal.
   d) Whole turn is the same as half turn then half turn.

III Let u, v, and w stand for any members of a set S, let i be
a special member of S, let * stand for combining any of
these in pairs, and let = mean is equivalent to or the
I same as. Using any of the sets you have studied such as
modular 4 arithmetic, give examples for each of the
following rules that are true:

a) $u*(v*w) = u*(w*v)$
b) $u*i = u$
c) $v*w \neq t$
d) $(u*v)*w = u*(v*w)$
e) $u*w = i$ and $v*w = i$

IV The following is a table of modular 6 arithmetic. The letters of the questions match those in the table where numbers have not been filled in. First answer the questions then fill in the answers in the spaces with the same letters.

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	a)	b)
1	e)	2	3	4	5	L)
2	f)	g)	4	5	m)	
3	h)	r)	k)	n)		
4	c)	j)	p)		3	
5	d)	q)				

1) a. $0 \oplus 4 =$ b. $0 \oplus 5 =$
c. $4 \oplus 0 =$ d. $5 \oplus 0 =$

2) e. $1 \oplus 0 =$ f. $2 \oplus 0 =$
g. $2 \oplus 1 =$ h. $3 \oplus 0 =$
r. $3 \oplus 1 =$ j. $4 \oplus 1 =$
k. $3 \oplus 2 =$

3) L. $1 \oplus 5 =$ m. $2 \oplus 4 =$
n. $3 \oplus 3 =$ p. $4 \oplus 2 =$
q. $5 \oplus 1 =$

4) Which of the following is true?

i) $5 \oplus 2 = 7$
ii) $5 \oplus 5 = 4$
iii) $4 \oplus 3 = 7$
iv) $5 \oplus 3 = 2$

5) The following is true:

$4 = 2 \oplus 2$
$5 \oplus 4 = (5 \oplus 2) \oplus 2$

Therefore, what is
$5 \oplus 4 = \quad ?$

V 1. Remember there are only four numbers in mod 4 arithmetic. Let E represent a number which is not one of those of mod 4 arithmetic. Let A and B be any of the four numbers of mod 4 arithmetic.
I Then we know that $A \oplus B \neq E$. Which of the following statements expresses the same idea?

a) $2 \oplus 3 \neq 1$
b) $3 \oplus 2 \neq 3$
c) $2 \oplus 3 \neq 5$
d) $3 \oplus 2 \neq 0$

2. Let the symbols A, B, C, and D represent the numbers of mod 4 arithmetic. Which of the following are always true for mod 4?

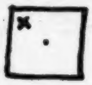
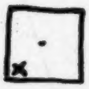
a) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
b) $A \oplus C = C \oplus B$
c) $A \oplus (C \oplus B) = (B \oplus C) \oplus A$
d) $(A \oplus A) \oplus D = C \oplus (A \oplus D)$

Give at least one example of each of the above that is true.

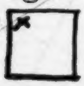
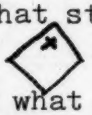
3. Let A represent a whole turn and B, C, and D represent the other motions of the happy face card. For each of the following give an equivalent happy face card motion: the . means 'followed by'.

   a) B.A = B, C = C.A, A.D = D
   b) B.B = A = C.C = D.D

VI The square spinning described below works the same way as the games you have been playing over the past few weeks such as happy face card moves, mod 4 arithmetic and □ , ○ , △ , and ▬ . That is, the rules of putting pairs of them together is the same.

A The moves in square spinning are different numbers of right angles, like one right angle (90°), two right angles (180°), and so on. The square has an X at the top left corner like this:  Spinning the square through right angles moves the X corner to other corners. For example, spinning the square about the center point in the plane of the paper through three right angles (270°) moves the X corner to the lower left like this: 

1. For the square spinning which of the following is true?

   a) a 180° spin then a 270° spin puts the X corner in the same place as a 270° spin then a 180° spin
   b) a 180° spin then another 180° spin puts the X corner in the same place as a 360° spin.
   c) doing any spin will move the X corner to a position. Following this by a 360° spin will leave the X corner in that position.
   d) It is possible to combine a number of right angle moves so that starting from this  you can get this  .

2. If you already know what a 180° spin then a 270° spin will give you. Then you could do 90° then 180° then 270° in two moves instead of three. You would do 90° then the result of 180° and 270° combined. Would the X corner end up in the same place if you combined the 90° and 180° into one move and then did the 270° spin?

VII The switch throwing described below works in the same way as the other games you have played such as happy face card flipping. That is, the rules of putting pairs of switch throws together are the same as putting pairs of moves from the other games together. Suppose we have two light switches. Starting with A both lights off we can switch the left, or we can switch the right, or we can switch both together, or we can switch neither. For example, starting with neither on, then switching the left turns the left light on. Then throwing the left switch again turns the left light off, so it amounts to the same thing as throwing neither switch. If this is to be compared to the motions of the rectangular card happy face,

1. Is there an equivalent to the idea that any motion followed by a 360° rotation is the same as just that motion alone? If so give at least two examples.

2. Does the order of turning a left switch and a right switch make any difference in what lights will be on in the end?

3. A rule of the light switching is as follows: switching the left and right together and then the left is the same as switching the left and then the right and left together. Give an example of the same rule for the happy face card motions.

4. If throwing a switch half way is considered as another move, which, if any, of the other moves combined would give the same result? Do you think throwing a switch half way can be a move if we want the rules of the switch throwing to be the same as the rules of the happy face card or mod 4 arithmetic?

5. A rule of light switching is left then right is the same as switching neither. Also left and right together then left and right together again is the same as neither. Which of the following rules from other games are these two rules like?

   a) Whole turn and half turn is half turn, and horizontal and vertical is half turn.
   b) In ordinary arithmetic 1 - 0 = 1, and 1 + 0 = 1
   c) ○ ↑ ▭ → □ , and △ ↑ △ → □ .
   d) 1 ⊕ 3 = 0, and 2 ⊕ 2 = 0.

VIII Use the symbols ',*,$,# to represent the motions or numbers in the ideas below. Let the ' symbol be like the whole turn or the 0 symbol. A dot like this . is S used between each pair of symbols that are combined with one another. Match the idea expressed on the left with the same idea expressed on the right in the new symbols.

1. $(2 \oplus 3) \oplus 2 = 2 \oplus (3 \oplus 2)$
2. Vertical then whole turn is the same as vertical
3. $\bigcirc \uparrow \square \rightarrow \square \uparrow \bigcirc$
4. $3 \oplus 1 = 0$
5. Vertical then vertical is not diagonal

a. (* . $) . * = * . ($. *)
b. * . ' ≠ $
c. (* . $) . # = # . #
d. # . # ≠ &
e. * . ' = *
f. * . $ = $. *
g. * . # = '

ARITHMETIC ACHIEVEMENT TEST

Name ....

1. Add

(a)	(b)
3647 8492 6386	6 7/8 8 9/12

2. Subtract

(a)	(b)
76032 54298	16 7/8 14 3/11

3. Multiply

(a) 725 x 905 (b) 9/11 x 6 3/8

4. Divide

(a) 29718 ÷ 127 (b) 8 1/3 ÷ 10 1/2

5. Add

46.75 + 4.98 + 28.4 + 654.88 =

6. Subtract

5.823 - 1.9999

7. Multiply

17.653 x 1.34

8. Divide

.288288 ÷ 2.91

9. Round off 382.254618 to the:

(a) nearest ten - ....

(b) nearest unit - ....

(c) nearest tenth - ....

(d) nearest hundredth - ....

(e) nearest thousandth - ....

10. Write the intersection set of the following sets:

A = (1/2, .6, 15, .25)

B = (1/25, .15, 3/5, 2)

11. Express the following numbers in scientific notation.

(a) 3,000,000,000 - ....

(b) 186,950,000 - ....

12. Perform the following operations:

(a) 74.28 x 1000 = (b) 3.874 ÷ 100 =

(c) 34.25 ÷ 1000 = (d) 100,000 x 1.0071 =

(e) 1000 x 3.0316 = (f) 5.139 ÷ 10 =

(g) 1.715 ÷ 100 = (h) 76.89 x .001 =

(i) 3.01 x 1035 = (j) 171 x .0001 =

13. Problems: Write open sentences for the following problems and solve them.

(a) The batting averages of the members of a baseball team were as follows:

.316
.308
.296
.288
.275

.271
.270
.265
.255
.242

.220
.220
.190
.186
.134

What was the team average?

(b) How many tiles each 2/3 of an inch long will fit in a groove 7 1/3 inches long?

(c) If one sheet of paper is 1/200 inch thick, how thick will a book be that contains 540 pages, if the covers are each 3/64 inches thick?

(d) A racing boat had a rated speed of 65 m.p.h. At that rate, how far could it go in 140 minutes?

(e) A farmer stated that he usually expected to lose 1/3 of his strawberry crop due to all causes. One year he lost 1/8 of his crop due to bad weather, 1/7 of his crop due to insects and 1/10 of his crop due to spoilage after picking. How did this particular year compare with his average year?

(f) How many sheets of paper .008" thick are there in a book which is 1.4" thick if each cover is .12" thick?

(g) In a certain stretch of mountain highway, 4.7 miles of highway are under repair. 1.8 miles of road requires blasting at a cost of $10,500.00 per mile. The remainder of the road requires grading at a rate of $6,500.00 per mile. The entire road requires finishing at a cost of $10.60 per yard. What is the cost of this section of road?

(h) There were 3,638 Canadian post offices in 1868. By 1958 there were 12,500. How many more Canadian post offices were there in 1958 than in 1868?

NAME .... Last First Middle

SCHOOL ....

AGE .... Years

GRADE ....

BOY GIRL (Circle One)

DATE .... Day Month Year

SK4: PART I

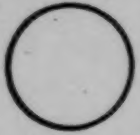
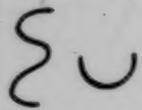
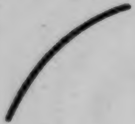
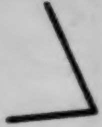
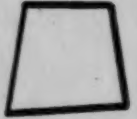
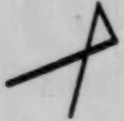
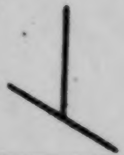
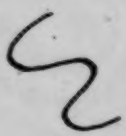
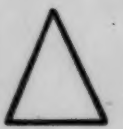
INSTRUCTIONS

There are 15 rows of figures in PART I of this test. In each row of figures, the first three figures (marked "EXAMPLES") all have some property in common. In the second group of three (marked "NOT EXAMPLES") in the row, none of the figures has this property. Your problem is to decide whether or not each of the figures under the question "ARE THESE EXAMPLES?" has the property. If it has the property (i.e., if it is an example), circle YES under that figure. If it does not have the property, circle NO.

Look at the first row of figures at the top of the next page. Each of the figures under "EXAMPLES" is made of curved lines only. None of the figures under "NOT EXAMPLES" has this property. Look under the question "ARE THESE EXAMPLES?" Figure 1 is not made of curved lines, so you should circle NO for figure 1. Since figure 2 is curved, you should circle YES for figure 2. Figure 3 is not made of curved lines so you should circle NO for figure 3.

The second row of figures deals with another property. After deciding what the property is by looking at the figures under "EXAMPLES" and under "NOT EXAMPLES," answer the question "ARE THESE EXAMPLES?" by circling YES or NO for figure 4; then for figure 5; and then for figure 6. Then go on to the rest of the rows of figures in PART I of the test.

IMPORTANT: One, two, or all three of the figures under "ARE THESE EXAMPLES?" may have the property (i.e., may be examples).

EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
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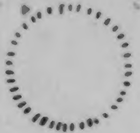
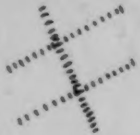
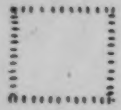
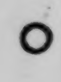
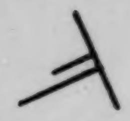
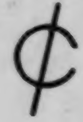
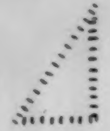
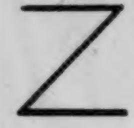
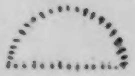
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
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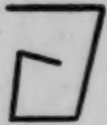
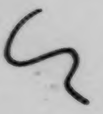
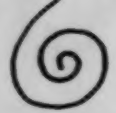
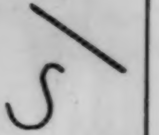
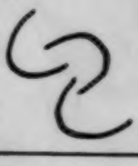
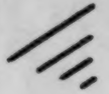
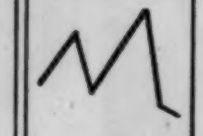
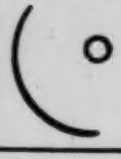
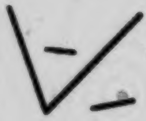
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
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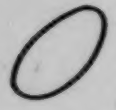
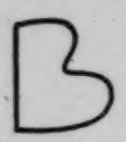
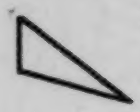
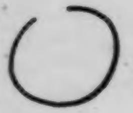
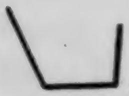
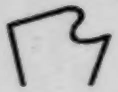
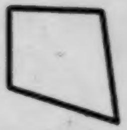
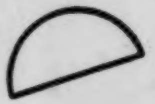
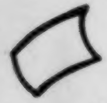
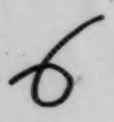
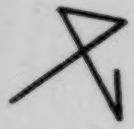
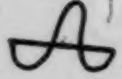
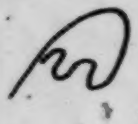
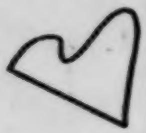
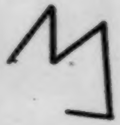
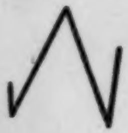
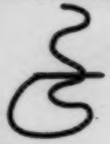
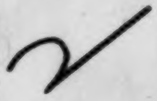
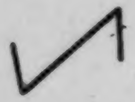
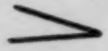
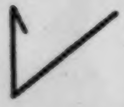
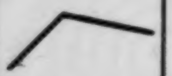
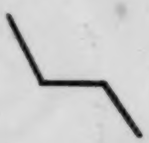
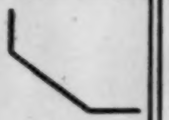
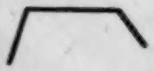
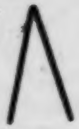
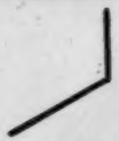
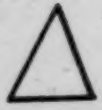
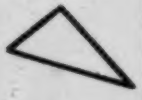
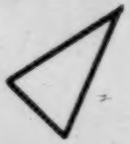
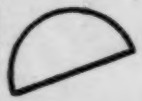
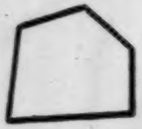
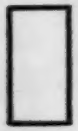
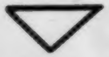
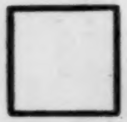
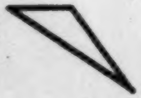
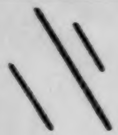
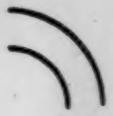
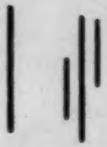
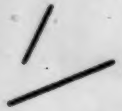
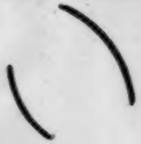
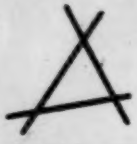
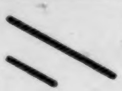
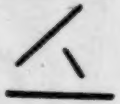
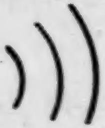
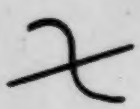
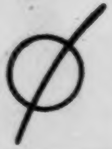
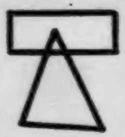
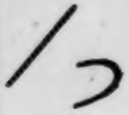
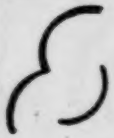
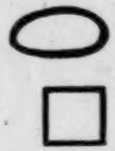
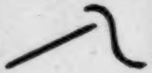
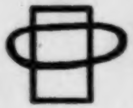
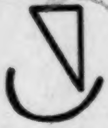
								
						10. YES NO	11. YES NO	12. YES NO

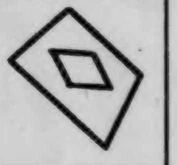
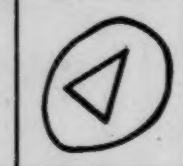
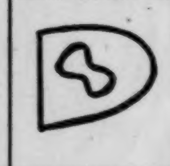
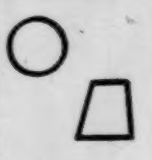
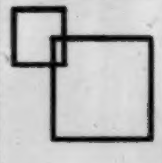
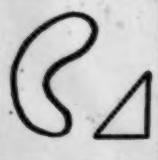
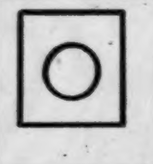
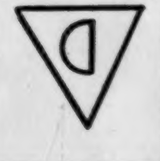
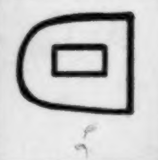
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						13. YES NO	14. YES NO	15. YES NO

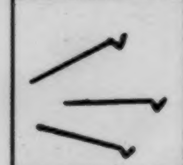
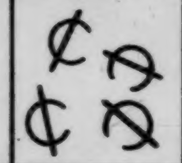
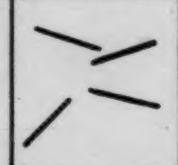
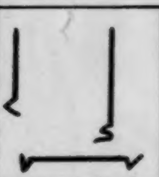
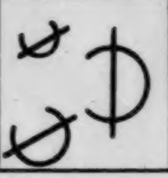
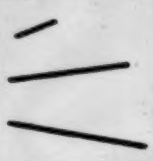
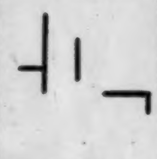
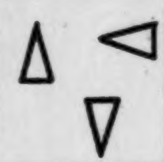
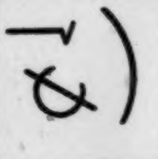
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						16. YES NO	17. YES NO	18. YES NO

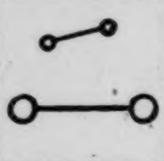
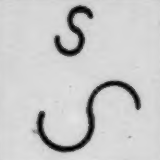
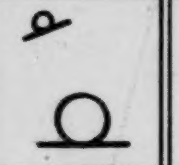
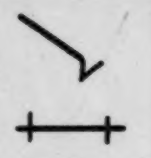
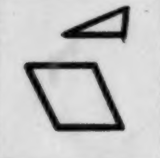
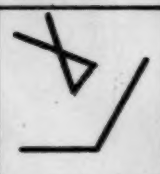
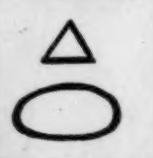
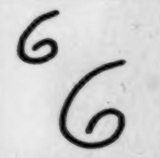
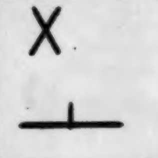
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						19. YES NO	20. YES NO	21. YES NO

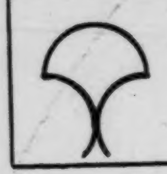
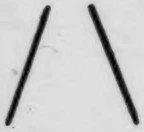
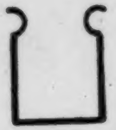
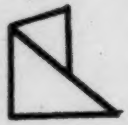
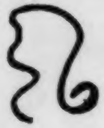
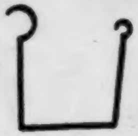
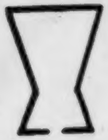
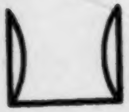
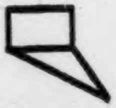
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						22. YES NO	23. YES NO	24. YES NO

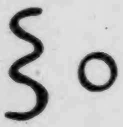
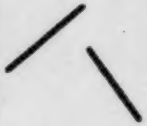
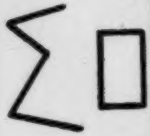
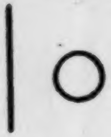
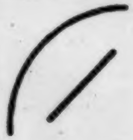
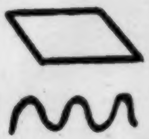
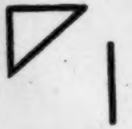
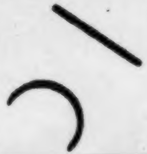
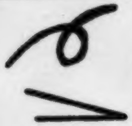
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						25. YES NO	26. YES NO	27. YES NO

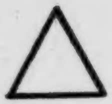
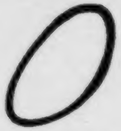
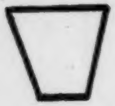
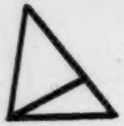
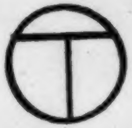
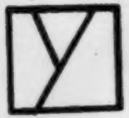
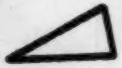
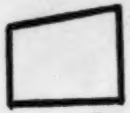
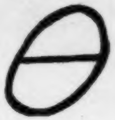
EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						28. YES NO	29. YES NO	30. YES NO

EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						31. YES NO	32. YES NO	33. YES NO

EXAMPLES			NOT EXAMPLES			ARE THESE EXAMPLES?		
								
						34. YES NO	35. YES NO	36. YES NO

E X A M P L E S			N O T E X A M P L E S			ARE THESE EXAMPLES?		
								
						37. YES NO	38. YES NO	39. YES NO

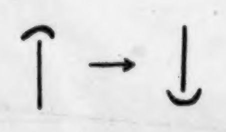
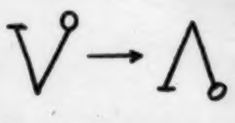
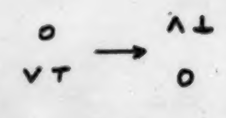
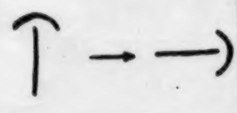
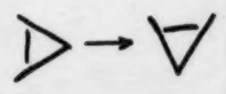
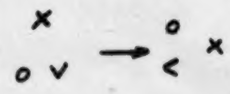
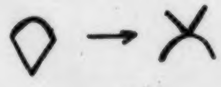
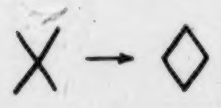
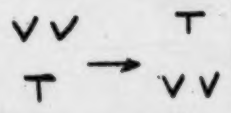
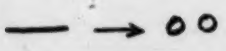
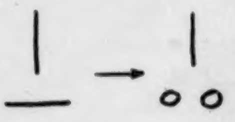
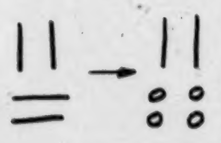
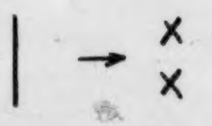
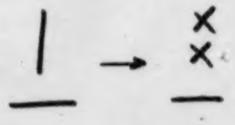
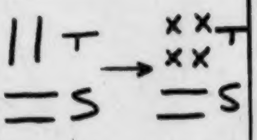
E X A M P L E S			N O T E X A M P L E S			ARE THESE EXAMPLES?		
								
						40. YES NO	41. YES NO	42. YES NO

E X A M P L E S			N O T E X A M P L E S			ARE THESE EXAMPLES?		
								
						43. YES NO	44. YES NO	45. YES NO

SK6: DEMONSTRATION SHEET

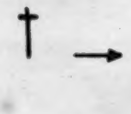
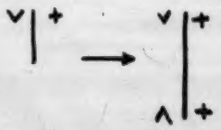
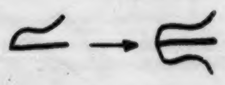
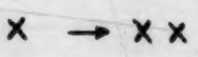
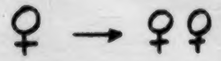
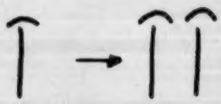
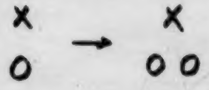
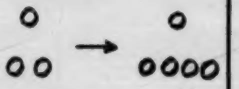
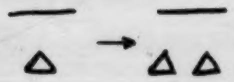
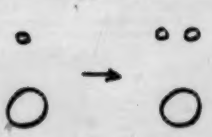
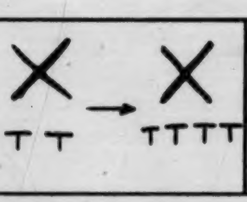
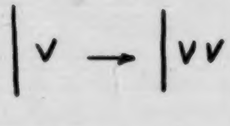
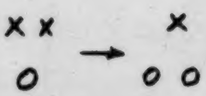
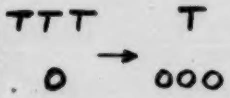
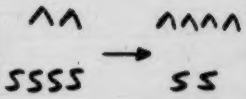
OPERATIONS A TO E

(OPERATIONS F TO J ARE ON THE NEXT PAGE)

Operation A			
Operation B			
Operation C			
Operation D			
Operation E			

SK6: DEMONSTRATION SHEET

OPERATIONS F TO J

Operation F			
Operation G			
Operation H			
Operation I			
Operation J			

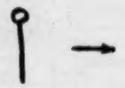
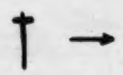
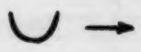
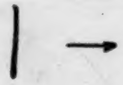
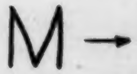
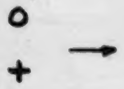
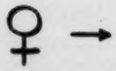
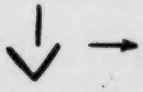
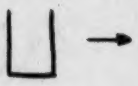
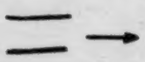
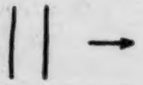
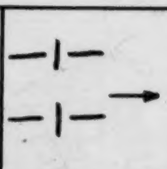
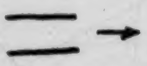
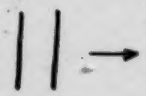
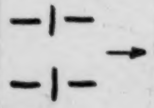
Page 1

NAME .... SCHOOL ....
Last First Middle

AGE .... Years GRADE .... BOY GIRL (Circle One) DATE .... Day Month Yr.

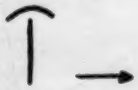
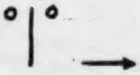
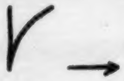
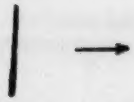
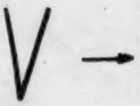
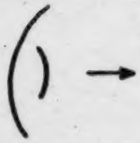
SK6: PART I

Find out the operations from the DEMONSTRATION SHEET, and fill in the answers in the blank spaces, just as you did on the PRACTICE SHEET.

Do Operation A on these.			
Do Operation B on these.			
Do Operation C on these.			
Do Operation D on these.			
Do Operation E on these.			

SK6: PART I

(CONTINUED)

Do Operation F on these.			
Do Operation G on these.			x oo →
Do Operation H on these.	S T →	o + →	O + →
Do Operation I on these.	o + →	O + →	
Do Operation J on these.	XX ooo →	T SS →	ΛΛΛ /// →

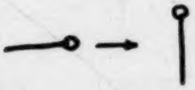
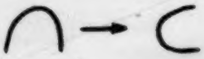
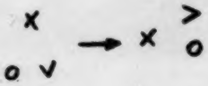
NAME .... SCHOOL ....
Last First Middle

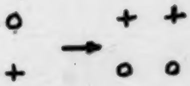
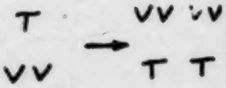
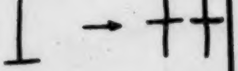
AGE .... GRADE .... BOY GIRL DATE ....
Years (Circle One) Day Month Yr.

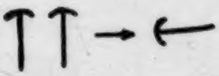
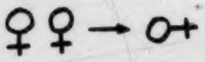
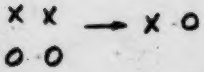
SK6: PART II

In PART II the problem is to combine the operations on the DEMONSTRATION SHEET, or to do them in reverse, or both. When combining operations, they are to be done in the order given (i.e., "Combine C and G" means "Do Operation C first and then do Operation G.")

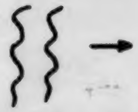
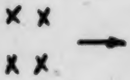
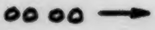
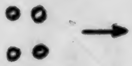
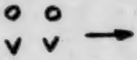
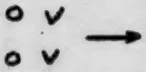
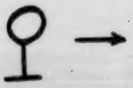
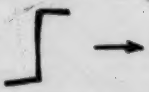
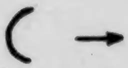
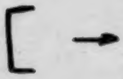
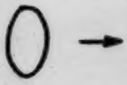
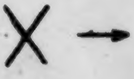
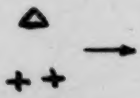
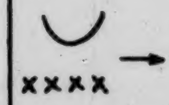
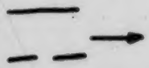
Look at the examples given below and then carry out the operations indicated on the following three pages.

EXAMPLE: Reverse B			

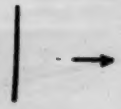
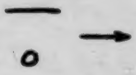
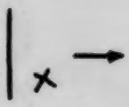
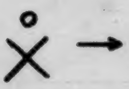
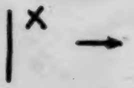
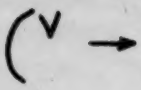
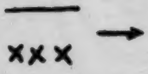
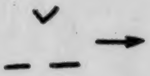
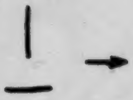
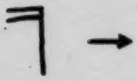
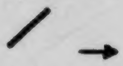
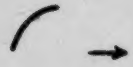
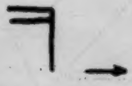
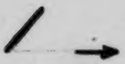
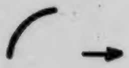
EXAMPLE: Combine C & G			

EXAMPLE: Reverse and Combine G & B			

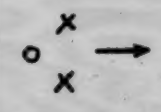
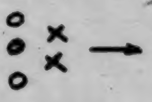
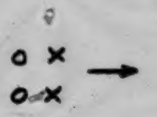
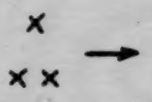
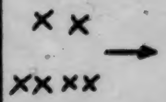
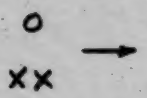
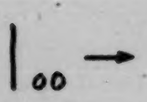
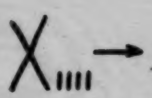
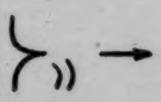
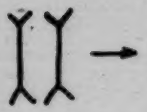
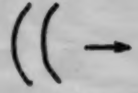
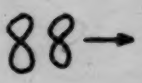
SK6: PART II

Reverse G			
Reverse D			
Reverse C			
Reverse F			
Reverse H			

SK6: PART II

Combine E & H			
Combine A & I			
Combine D & J			
Combine B & F			
Combine F & B			

SK6: PART II

Reverse and Combine B & J			
Reverse and Combine H & E			
Reverse and Combine A & I			
Reverse and Combine F & G			
Reverse and Combine A & C